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MOMENT DISTRIBUTION APPLIED TO TRUSSED BENTS

By

Henry T. Thornton, Jr.

Thesis submitted to the Graduate Faculty of the

Virginia Polytechnic Institute

in candidacy for the degree of

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in

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ABSTRACT

A method is devised by which moment distribution may be applied in the ordinary manner to trussed bents and by which the effects of truss thrust may be included. A single point at the end of the truss where moment and thrust can be varied independently of each other is located and defined as the "neutral point" of the truss. A substitute column-truss attachment is devised to give a concentrated equivalent joint between the column and the truss at the "neutral point." These innovations provide a substitute structure for the purpose of analysis. The method involves the distribution of moments only with the equivalent joints locked against translation, and subsequent corrections for the thrusts in a manner analogous to the simultaneous equation method of shear correction in multistory building frames. Equations to express the necessary truss constants are developed in general terms of the truss stress summations and truss dimensions. A substitute column is described and the necessary column constants are derived and expressed in general form. Numerical examples of the determination of the truss constants are given. Ten numerical examples of the application of the method are presented and explained in detail.

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V. NOMENCLATURE

A	an area when appearing in L/AE or AE
E	Young's modulus of elasticity
FEM	abbreviation for fixed end moment
FET	abbreviation for fixed end thrust
H	horizontal reaction or force
I	moment of inertia
L	individual truss member length only when appearing in L/AE
L	total truss length otherwise
M	a moment
F^M	a fixed end moment
S_p	individual truss member stress (in pounds) due to external loads at the truss panel points
S	individual truss member stress (nondimensional) due to a load of unity at a specified point, always with a capatilized subscript H, X, Y, or Z to denote the origin of the stress
T	a thrust
F^T	a fixed end thrust
V	a vertical reaction
K_c	column moment stiffness
K_s	moment due to column shear stiffness
K_t	truss thrust stiffness
K_T	truss moment stiffness
h	a column length dimension

l truss panel length

r_c column moment carryover factor (additional capitalized subscripts, if any, refer to the direction of carryover)

r truss moment carryover factor

δ distance from the upper end panel point of the truss to the "neutral point"

φ an angular rotation

Δ a translation

γ, β truss distribution factors for moment

$\sum S_i S_j$ shorthand notation for the summation of the individual products or squares of the individual truss member stresses due to a load of unity, the product or square being multiplied by the L/AE of the respective member. That is, $\sum S_i S_j$ always implies $\sum S_i S_j \frac{L}{AE}$, in this presentation if the L/AE term is not specifically indicated in the summation. $i = X,Y,Z$; $j = X,Y,Z$

VI. INTRODUCTION

The application of moment distribution to trussed bents is not entirely new, having been applied with varying degrees of success by at least one author[1] and several previous investigators [2,3,4,5]. The principle difficulty in the application is that, in general, when the end of a truss is rotated resultant end thrusts are created, similar to the action of arched or curved beams of solid cross section. Another difficulty is that for a truss the mathematical expression of the constants required for moment distribution is not relatively simple as in the case of solid beams.

Maugh[1] bypassed the thrust problem by neglecting the effects of thrust entirely. Later investigations[2,3,4,5] have shown that although this omission is not particularly serious in the case of a symmetrical bent with relatively flexible pin ended columns, the omission of the effects of thrusts in bents with fixed column bases, relatively stiff column and/or unusual multibay structures can result in stresses which cannot be neglected.

Three investigators[2,3,4] have included the effects of thrusts and have used a procedure of simultaneous moment and thrust distribution about a joint, converting from one to the other between cycles of distribution. The convergence by this method is very slow. In addition, the column moment and shear stiffnesses are based on an equivalent length column in order to obtain a concentrated point of attachment to the truss about which the moments and thrusts could be balanced.

Another investigator[5] used the "neutral point" concept devised by Cross and Morgan[6] for arched type structures. This again requires, in effect, the simultaneous distribution of moments and thrusts although the convergence is improved greatly. The solutions obtained are within engineering design accuracy but the method is long and involved.

The most commonly used method for obtaining reliably accurate solutions is by the method of Least Work. Although this is relatively simple for a single bay structure, for multibay structures the method becomes quite involved, the actual labor increasing somewhat as the square of the number of redundants in the structure.

The present investigation was undertaken to obtain a method which would give reasonably accurate results and which could be easily applied. The method presented herein preserves the true deflection geometry of both the truss and the column and furthermore converges to the "exact solution," as obtained by the method of Least Work. It involves the distribution of moments only with the joints locked against translation, and subsequent corrections for the thrusts by a method analogous to that of the simultaneous equation method of shear correction for sidesway in multistory building frames. The convergence of the moment distribution is relatively rapid. It is believed that the method is easier to understand and is easier to apply than any of the previous methods devised, excepting, of course, the erroneous one where thrusts are completely omitted.

It is assumed that the reader is familiar with the basic methods of structural analysis, including moment distribution and the method of Least Work.

VII. THE ASSUMPTIONS

The method presented herein is based on the following assumptions:

1. A linear stress-strain relationship of the material exists within the magnitude of stress and strain considered.
2. The principle of superposition is valid.
3. All truss connections are considered pinned and the lines of action of the members are coincidental with the pins.
4. All truss loadings are concentrated at the panel points with no local member bending.
5. The effects of axial loads on the columns are neglected.
6. The effect of column shear deformation is neglected.

VIII. DEFINITIONS

The following definitions are those generally accepted and used in the application of moment distribution and were taken from the text "Indeterminate Structural Analysis."[7] As such, they are not necessarily an essential part of this presentation and are included only to remind the reader and in order that they need not be reproduced in the body of the material.

1. Absolute moment stiffness is the value of the moment, applied at a simply supported end of a member, necessary to produce a rotation of 1 radian of this simply supported end, no translation of either end being permitted, and the far end being restrained or fixed.
2. The distribution factor for any member at a joint is equal to the stiffness of the member divided by the sum of the stiffness of all members at the joint.
3. The carryover factor is that factor by which the developed moment at the rotated end of a member may be multiplied (the other end being fixed or restrained) to give the induced moment at the fixed or restrained end.
4. The absolute shear stiffness is defined as the shear in a member that results when one end of the member is displaced a unit distance with respect to the other end, the displacement being in a direction perpendicular to the member axis, and the rotation of any rigid joint at the ends of the member being prevented.
5. The absolute moment due to shear stiffness is that moment induced at the ends (or end, if the member is pinned at one end) of a member

when one end of the member is displaced a unit distance with respect to the other end, the displacement being a direction perpendicular to the member axis, and the rotation of any rigid joint at the ends of the member being prevented.

The following definitions are intended to apply specifically to trusses in this presentation.

1. Absolute thrust stiffness is that value of thrust, applied at a simply supported end of a truss, necessary to produce a horizontal (axial) translation of one unit at that end, no rotation of either end being permitted, and the far end being fixed against translation. (Note that the requirement of no end rotation may or may not require an applied moment at the end.)

2. The thrust carryover factor is that factor by which the developed thrust at the translated end of a truss may be multiplied (the other end being fixed) to give the induced thrust at the far end. (The value of the thrust carryover factor in this presentation is always negative unity. That is, the thrust at the far end is equal and opposite to the developed thrust.)

IX. DEVELOPMENT OF THE METHOD

A. The Least Work solution of a trussed bent

Consider the Least Work method of solution of the generalized trussed bent shown in figure 1(a). For simplicity and convenience hinged column bases have been selected. Because the Least Work method of solution is well known and is available in numerous texts, various steps and minor details have been omitted here. The total strain energy in the structure is given by

$$U = \frac{1}{2}\sum (S_{P_i} + H_A S_H)^2 \frac{L}{AE} + \frac{1}{2}\int_0^h \frac{M_1^2}{EI_1}\,ds + \frac{1}{2}\int_0^{h_4} \frac{M_2^2}{EI_2}\,ds$$

Where

S_{P_i} = Truss member stress due to a generalized load.

S_H = Truss member stress due to $H_A = 1$ unit.

M_i = Bending moment in column "i" due to $H_A = 1$ unit. $i = 1,2$

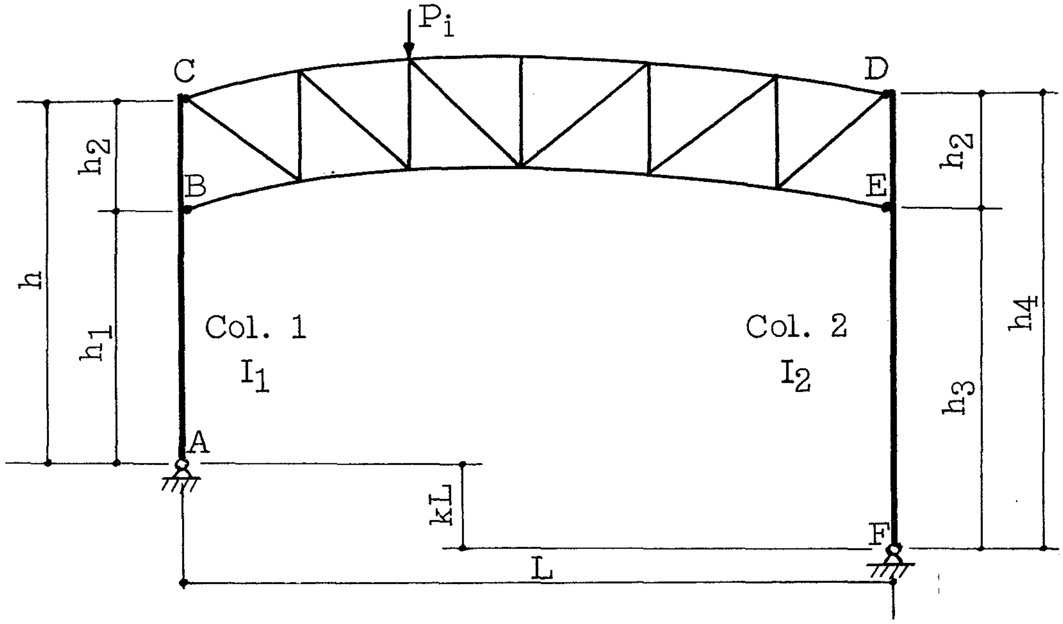
The redundant horizontal reaction H_A is obtained by applying Castigliano's Second Theorem and solving for H_A.

$$\frac{\partial U}{\partial H_A} = \sum (S_{P_i} + H_A S_H) S_H \frac{L}{AE} + \int_0^h \frac{M_1}{EI_1}\frac{\partial M_1}{\partial H_A}\,ds + \int_0^{h_4} \frac{M_2}{EI_2}\frac{\partial M_2}{\partial H_A}\,ds = 0$$

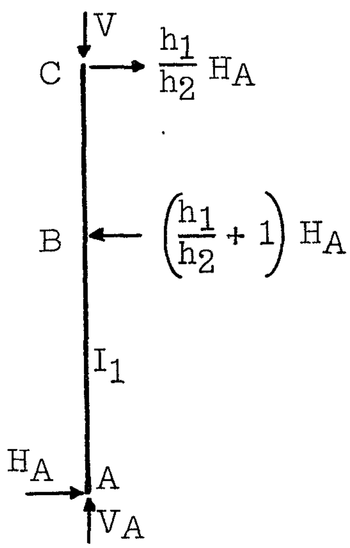
Recognizing that the moment expressions are explicit in H_A and that H_A may be factored from the integral terms

$$H_A = \frac{-\sum S_{P_i} S_H \frac{L}{AE}}{\sum S_H^2 \frac{L}{AE} + \int_0^h \frac{m_1^2}{EI_1}\,ds + \int_0^{h_4} \frac{m_2^2}{EI_2}\,ds}$$

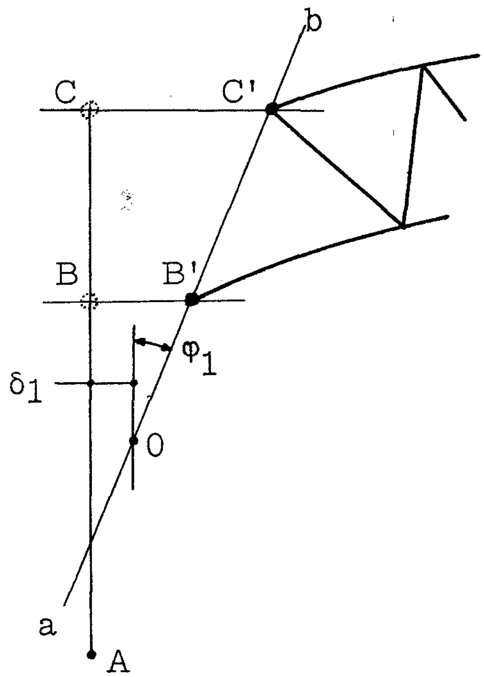
where m is now a function only of the column lengths and loads.



(a) Elevation of bent.



(b) Column force diagram.



(c) Truss end deflection diagram.

Figure 1.- Single bay bent and elemental details.

The redundant reaction H_A obtained by this method is considered the "exact solution" within the limits of the assumptions previously stated. With H_A known the bent becomes statically determinate.

Now consider column 1 as a free body as shown in figure 1(b). From the column M/EI diagram the relative shape of the deformed column may be computed but the absolute horizontal deflections of points C and B relative to point A cannot be determined from this information alone. The absolute numerical displacements may be found by various techniques using the entire structure but this would require considerable additional work. However, numerical quantities are not necessary to form a qualitative description of the end movement.

The ends of the truss have undergone some horizontal displacement from C to C' and from B to B' as shown in figure 1(c). This deflection pattern may be expressed uniquely as a rotation of a straight line through the truss end points about some point O and a horizontal translation of point O in relation to point A. The location of point O along the line ab is completely arbitrary.

If the column is replaced by a rigid bar connecting the points O, B', and C', the end of the truss could be held in its deflected position by the application of a moment and a thrust applied at point O. In general, the magnitudes of each will vary depending upon the location of point O and the rotation φ_1.

B. The truss and its end deflections

In the preceding section the deflections of the end of the truss as a part of the whole structure was examined. Now consider a generalized truss as shown in figure 2(a) where X, Y, and Z represent general end loads at the respective locations. In the following discussion where the Least Work method of solution for unknowns is used

S_X = truss member stress due to $X = 1$ unit

S_Y = truss member stress due to $Y = 1$ unit

S_Z = truss member stress due to $Z = 1$ unit

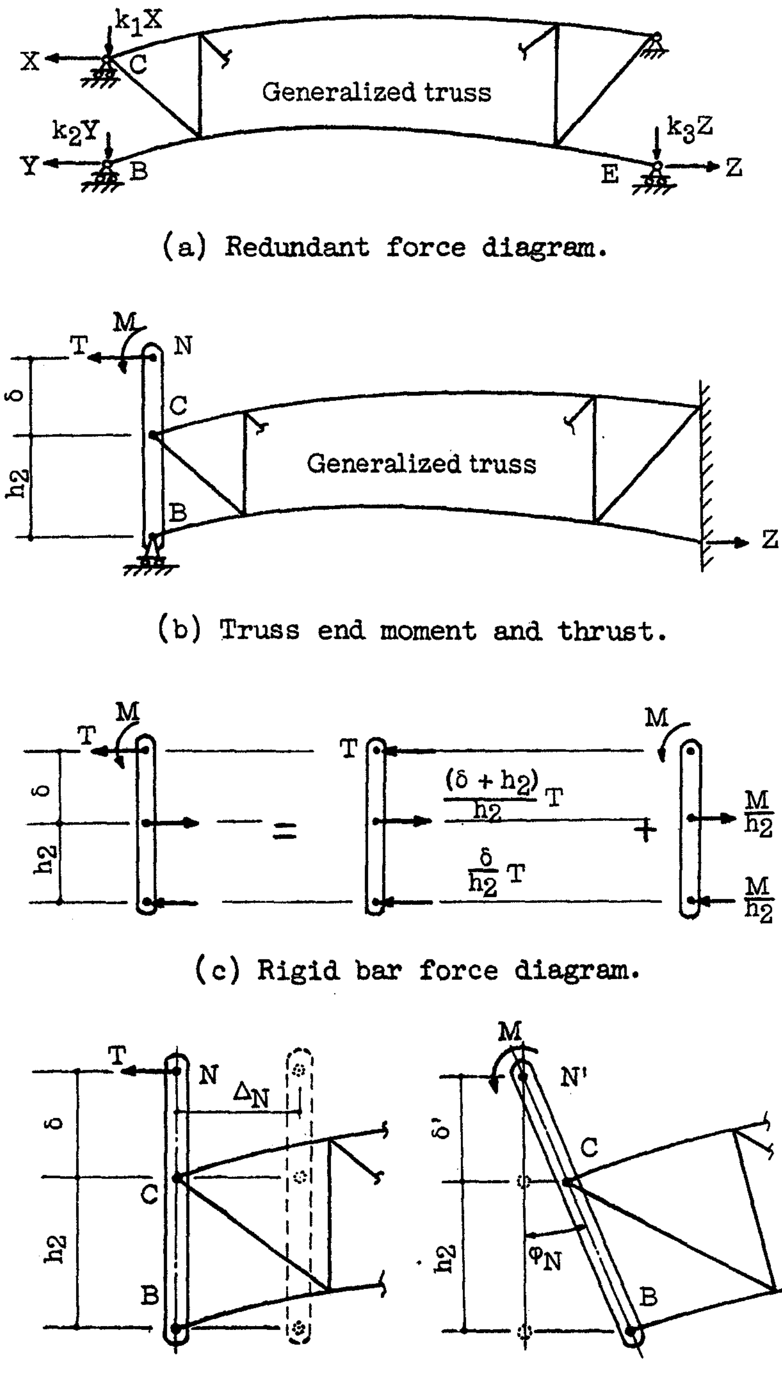
The applied forces X, Y, and Z are positive as shown.

The total strain energy in the system is given by

$$U = \frac{1}{2}\sum (XS_X + YS_Y + ZS_Z)^2 \frac{L}{AE} \qquad (1.01)$$

By attaching a rigid bar to the left end as shown in figure 2(b) a force system applied at N can be transmitted to the truss. The resultant forces at the chord ends are shown in figure 2(c), the applied forces to the truss end being of opposite sign. Being interested in the deflections of the left end only and also in order to conform to certain definitions which will be required later, let the right end be fixed as shown in figure 2(b). Substituting into equation (1.01) for X and Y the values in terms of T and M from figure 2(c), the strain energy expression becomes

$$U = \frac{1}{2}\sum \left\{\left[\frac{(8 + h_2)}{h_2} T + \frac{M}{h_2}\right]S_X - \left[\frac{8}{h_2} T + \frac{M}{h_2}\right]S_Y + ZS_Z\right\}^2 \frac{L}{AE} \qquad (1.02)$$



(a) Redundant force diagram.

(b) Truss end moment and thrust.

(c) Rigid bar force diagram.

(d) End translation. (e) End rotation.

Figure 2.- Truss end force systems and deformations.

The end deflections of the left end may be described completely from the equations obtained by applying Castigliano's First Theorem

$$\frac{\partial U}{\partial T} = \Delta_N \tag{1.03}$$

$$\frac{\partial U}{\partial M} = \varphi_N \tag{1.04}$$

$$\frac{\partial U}{\partial Z} = 0 \tag{1.05}$$

Equation (1.05) is used to eliminate Z from the other two equations.

There exists some point N at which an applied thrust T will produce a horizontal deflection Δ_N with no end rotation φ_N and no resultant moment M, as shown in figure 2(d). The location of this point may be found by setting

$$\frac{\partial U}{\partial M} = \sum\left\{\left[\frac{(\delta + h_2)}{h_2} T + \frac{M}{h_2}\right] S_X - \left[\frac{\delta}{h_2} T + \frac{M}{h_2}\right] S_Y + Z S_Z\right\}\left(\frac{S_X}{h_2} - \frac{S_Y}{h_2}\right)\frac{L}{AE} = 0 \tag{1.06}$$

$$\frac{\partial U}{\partial Z} = \sum\left\{\left[\frac{(\delta + h_2)}{h_2} T + \frac{M}{h_2}\right] S_X - \left[\frac{\delta}{h_2} T + \frac{M}{h_2}\right] S_Y + Z S_Z\right\} S_Z \frac{L}{AE} = 0 \tag{1.07}$$

The moment M has been specified as zero. Therefore it vanishes from equations (1.06) and (1.07). From equation (1.07) for $M = 0$

$$Z = -\frac{(\delta + h_2)}{h_2} T \frac{\sum S_X S_Z \frac{L}{AE}}{\sum S_Z^2 \frac{L}{AE}} + \frac{\delta}{h_2} T \frac{\sum S_Y S_Z \frac{L}{AE}}{\sum S_Z^2 \frac{L}{AE}}$$

To simplify the expression of the equations containing the summations of the truss member stresses it will be convenient to imply the inclusion of the term L/AE in the summation where the summation symbol is applied to individual terms. That is

$$\sum S_X^2 \quad \text{implies} \quad \sum S_X^2 \frac{L}{AE}$$

From equation (1.06), substituting for Z, and solving for δ

$$\delta = \frac{h_2\left\{-\sum S_X^2 + \sum S_X S_Y + \dfrac{\sum S_X S_Z}{\sum S_Z^2}\left(\sum S_X S_Z - \sum S_Y S_Z\right)\right\}}{\sum S_X^2 - 2\sum S_X S_Y + \sum S_Y^2 - \dfrac{1}{\sum S_Z^2}\left[\sum S_X S_Z - \sum S_Y S_Z\right]^2} \tag{1.08}$$

There exists also some point, say N', at which a moment may be applied which will produce only a rotation ϕ_N', with no translation Δ_N', and no resultant thrust T, as shown in figure 2(e). From statics it is obvious that the application of the moment at any point along the bar will produce the same resultant forces at B and C. However, the deflections of points B and C under this loading will define a point N' at which no translation occurs. To locate this point use equation (1.07)

$$\frac{\partial U}{\partial Z} = 0 \tag{1.07}$$

and set

$$\frac{\partial U}{\partial T} = \sum\left\{\left[\frac{(\delta' + h_2)}{h_2}T + \frac{M}{h_2}\right]S_X - \left[\frac{\delta'}{h_2}T + \frac{M}{h_2}\right]S_Y + ZS_Z\right\}\left\{\frac{(\delta' + h_2)}{h_2}S_X - \frac{\delta'}{h_2}S_Y\right\}\frac{L}{AE} = 0 \tag{1.09}$$

The thrust T has been specified as zero. Therefore, it vanishes from both equations. Then from equation (1.07) for $T = 0$

$$Z = -\frac{M}{h_2}\left[\frac{\sum S_X S_Z - \sum S_Y S_Z}{\sum S_Z^2}\right]$$

Substituting for Z in equation (1.09) and solving for δ'

$$\delta' \quad \frac{h_2\left\{-\sum S_X^2 + \sum S_X S_Y + \frac{\sum S_X S_Z}{\sum S_Z^2}\left(\sum S_X S_Z - \sum S_Y S_Z\right)\right\}}{\sum S_X^2 - 2\sum S_X S_Y + \sum S_Y^2 - \frac{1}{\sum S_Z^2}\left[\sum S_X S_Z - \sum S_Y S_Z\right]^2}$$

This equation is identical to equation (1.08). Therefore, the points N and N' are identical, and $\delta' = \delta$.

It is therefore established that any displacement of the points C and B due to any forces X and Y can be expressed uniquely by the application of the force system T and M acting at point N, and also conversely so. Furthermore, T and M are independent of each other, and the order of application is immaterial.

Herein lies the basic premise of the proposed method for applying moment distribution; for at the point N the moment may be varied without changing the thrust, and the thrust may be varied without changing the moment.

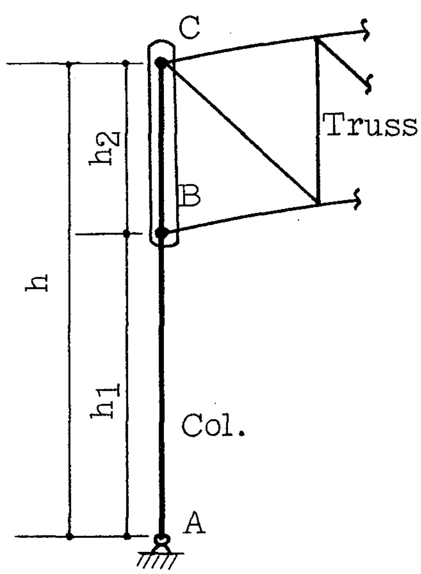
Because the point N obviously may not coincide with any part of a column attached to the truss, the next step is to examine the column-truss attachment.

C. The substitute column

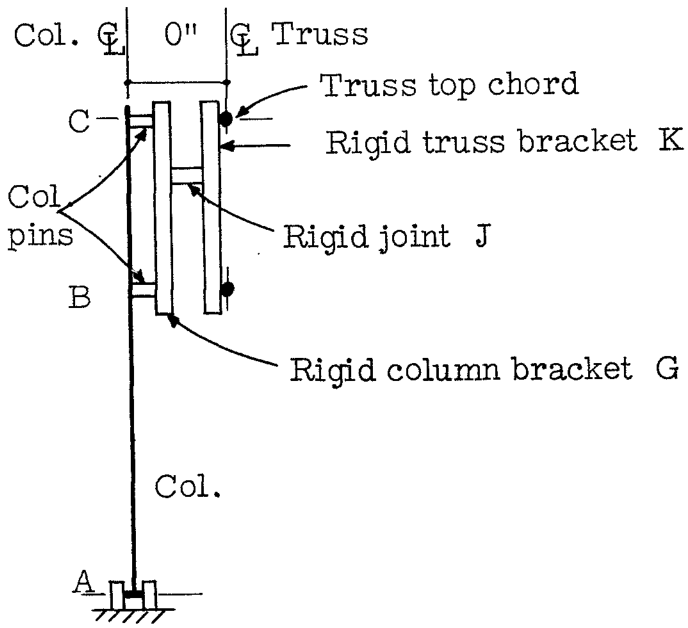
The truss attachment to the column consists of two pinned joints, B and C, figure 1(a), the column being continuous between A and C. The problem is to obtain a concentrated point of attachment on the column or an extension of the column at which a moment and horizontal force (thrust or shear) may be applied and yet make the column behave and deform as if it were attached to the truss at points B and C.

Both conditions are satisfied by the scheme illustrated in figures 3(a), 3(b), and 3(c). A rigid bar G is attached to the column by pins at the corresponding points of attachment of the truss. Another rigid bar K is attached to the truss at the end-panel points. Both the column and the end-panel points of the truss are free to rotate at these pinned joints relative to their respective rigid bars. The rigid truss bar, or bracket, and the rigid column bar, or bracket, are connected by a single concentrated rigid joint J. The rigid joint J prevents relative rotation between the two bars. Through this joint J, moment and thrust are transferred between the column and the truss. All force systems used lie in the single plane of the bent and columns.

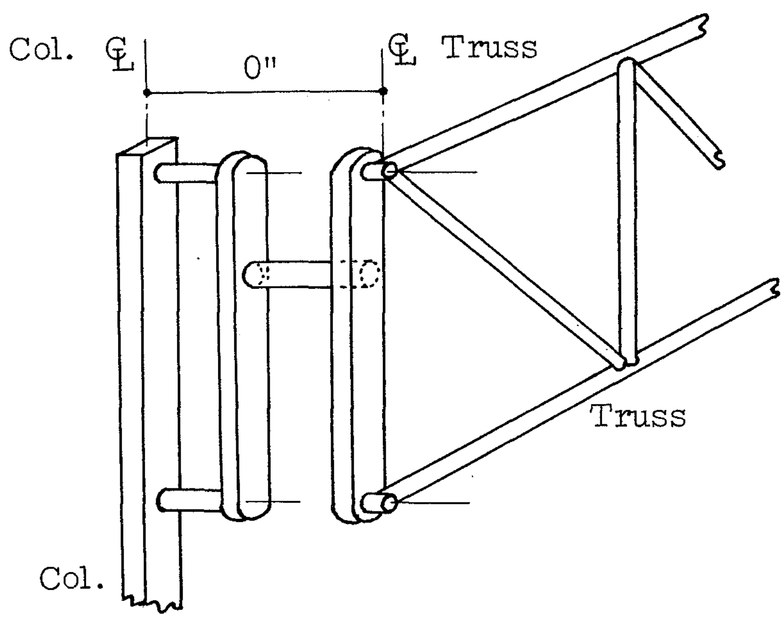
It is obvious that since the joint J and the brackets are all rigid, that for any end rotation or translation, the points on the column corresponding to the points of truss attachment will coincide with the end-panel points of the truss itself. Furthermore, by statics, it is evident as shown in figures 4(a), 4(b), 4(c), and 4(d) that the force



(a) Partial elevation of bent.

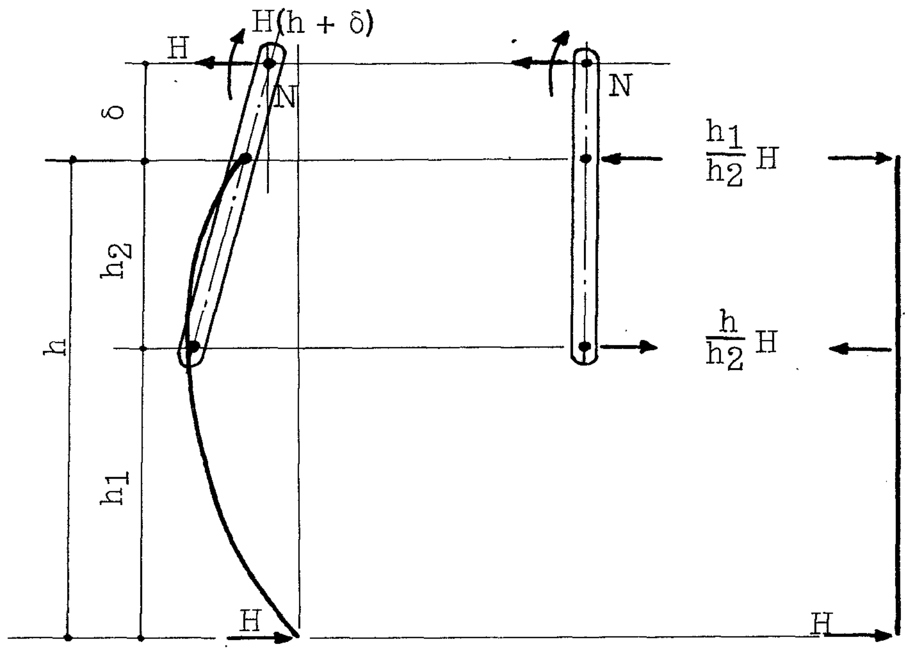


(b) Bracket schematic, end view.

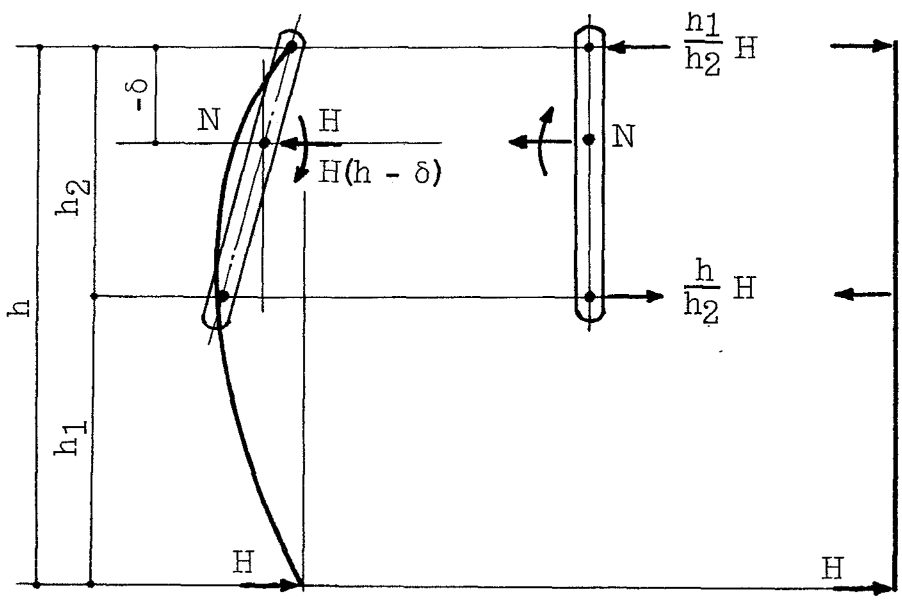


(c) Bracket perspective.

Figure 3.- Substititue column-truss attachment.



(a) Substitute column force system. (b) Force system in real column.



(c) Substitute column force system. (d) Force system in real column.

Figure 4.- Column force diagrams.

system existing in the real column-truss attachment is identical to that existing in the substitute column-truss attachment.

In order to gain advantage of the special truss properties found in the preceding section, it is necessary only to define the location of the rigid joint J as coinciding with the "neutral point" N of the truss. Of course the substitute column-truss attachment is imaginary and useful for the purpose of analysis only. The only remaining requirement is that the several stiffness factors and the moments accompanying unit translations for the column must now be defined relative to the "neutral point" N rather than at some point on the column. This is illustrated completely in section XI.

The point N is here defined to be the "neutral point" of the truss. It is the equivalent joint at which moments can be balanced in applying moment distribution to a trussed bent.

X. THE TRUSS CONSTANTS

A. General derivations

The necessary truss constants required for the application of moment distribution may be obtained in part from the equations developed in section IX, B. All of the constants are defined relative to the "neutral point," point N. The following derivations are in general terms into which numerical values may be substituted for the particular truss.

1. Moment stiffness.

The moment stiffness K_T may be obtained from equations (1.04) and (1.05). The resultant thrust T has been shown to be zero, leaving

$$Z = -\frac{M_N}{h_2}\left[\frac{\sum S_X S_Z - \sum S_Y S_Z}{\sum S_Z^2}\right]$$

$$\sum\left(\frac{M_N}{h_2} S_X - \frac{M_N}{h_2} S_Y + Z S_Z\right)\left(\frac{S_X}{h_2} - \frac{S_Y}{h_2}\right)\frac{L}{AE} = \varphi_N$$

From which, setting $\varphi_N = 1$ radian;

$$M_N\Big|_{\varphi_N=1} = K_T = \frac{h_2^2}{\left(\sum S_X^2 - 2\sum S_X S_Y + \sum S_Y^2\right) - \frac{1}{\sum S_Z^2}\left[\sum S_X S_Z - \sum S_Y S_Z\right]^2} \qquad (1.11)$$

The units of K_T are length times force per radian. Again it should be noted that in all of these equations $\sum S_i S_j$ implies $\sum S_i S_j \frac{L}{AE}$.

2. Carryover factor.

The moment at the right end of the truss is $h_2 Z$. Therefore the carryover factor is

$$r = \frac{h_2 Z}{M_N}\Bigg|_{\varphi_N=1} = -\frac{\sum S_X S_Z - \sum S_Y S_Z}{\sum S_Z^2} \tag{1.12}$$

The carryover factor is nondimensional.

3. Thrust stiffness.

The thrust stiffness K_t may be obtained from equations (1.03) and (1.05). The resultant moment has been shown to be zero, leaving

$$Z = -\frac{T}{h_2}(\delta + h_2)\frac{\sum S_X S_Z}{\sum S_Z^2} + \frac{T}{h_2}\delta\frac{\sum S_Y S_Z}{\sum S_Z^2}$$

$$\sum\left[\frac{T}{h_2}(\delta + h_2)S_X - \frac{T}{h_2}\delta S_Y + Z S_Z\right]\left[\frac{(\delta + h_2)}{h_2}S_X - \frac{\delta}{h_2}S_Y\right]\frac{L}{AE} = \Delta_N$$

From which, setting $\Delta_N = 1$ unit

$$T\Big|_{\Delta_N=1} = K_t =$$

$$\frac{(h_2)^2}{(\delta + h_2)^2\sum S_X^2 - 2\delta(\delta + h_2)\sum S_X S_Y + \delta^2\sum S_Y^2 - \dfrac{\left[(\delta + h_2)\sum S_X S_Z - \delta\sum S_Y S_Z\right]^2}{\sum S_Z^2}} \tag{1.13}$$

The units of K_t are force per unit length.

4. Fixed end moment and thrust.

The determination of fixed end moment and thrust requires another stress calculation for the truss loading. These stresses are computed from the truss as shown in figure 5(a). Using the stresses from a figure of the type in figure 2(a) also the strain energy in the truss becomes:

$$U = \frac{1}{2}\sum (Sp + XS_X + YS_Y + ZS_Z)^2 \frac{L}{AE} \qquad (1.14)$$

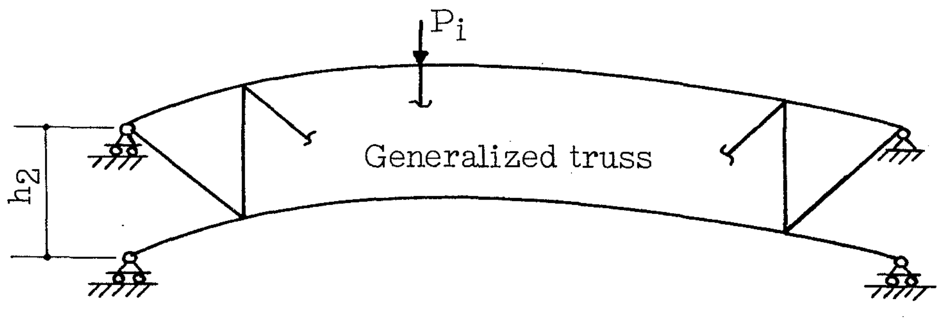
Where Sp = member stress due to the external loads.

Equations in the three unknown reactions X, Y, and Z are obtained by applying Castigliano's second theorem.

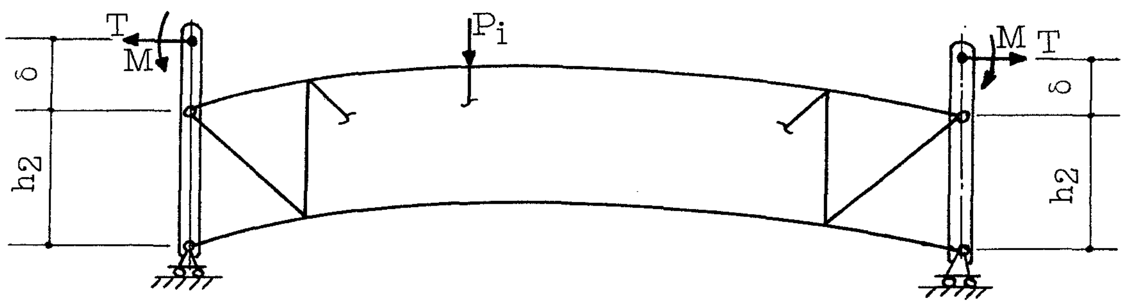
$$\frac{\partial U}{\partial X} = 0; \frac{\partial U}{\partial Y} = 0; \frac{\partial U}{\partial Z} = 0 \qquad (1.15)$$

The simultaneous solution of these equations gives the end reactions. These end reactions may be converted by statics into a moment and a thrust acting at point N as shown in figure 5(b). This is the fixed end moment and fixed end thrust for the particular truss loading. For these reactions it is desirable to use numerical values for the particular truss because it is not feasible to express the solution in generalized terms. The fixed end moment and thrust may also be found by using figures 5(a) and 5(b). However, the previous method is generally more convenient.

The following sections B and C provide numerical illustrations of the determination of the required truss constants. In general, the location of the "neutral point" will be between the top and bottom end-panel points for a horizontal parallel chord truss and above the upper end-panel point for a Fink truss. In either of the above, equation (1.08) will give the

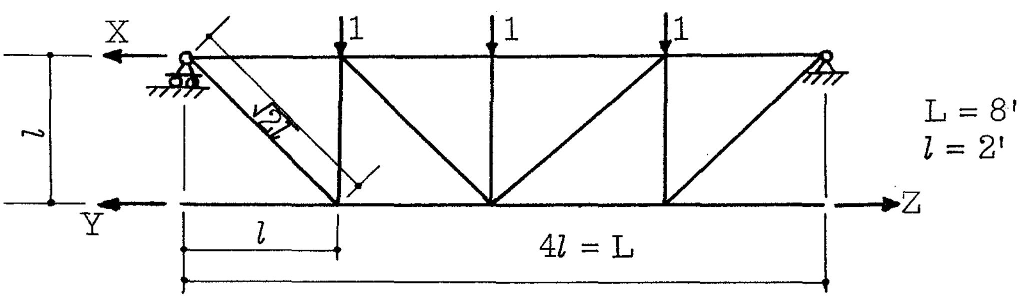


(a) Truss diagram for vertical load stresses.

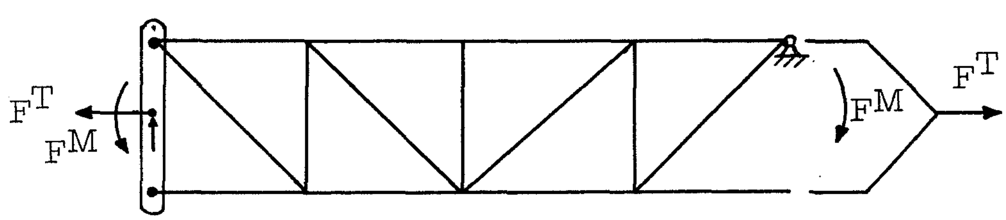


(b) Fixed end moment and thrust.

Figure 5.- Truss load diagrams.



(a) Load system and truss details.



(b) Alternate expressions of FEM and FET.

Figure 6.- Four panel Pratt truss.

correct sign for δ: positive for N above, and negative for N below, the upper end-panel point of the truss. No study has been made here to determine for what type of truss if any the "neutral point" would be located below the bottom end-panel point. If such should occur, the equation (1.08) should give the correct value for δ. However, a reevaluation of the derivation should be made.

It can be shown that in a horizontal parallel chord Pratt truss with constant area chords, the "neutral point" is independent of the web members and is located at the center of gravity axis of the top and bottom chords.

B. The parallel chord truss - a particular case

A highly simplified truss as shown in figure 6(a) has been selected to illustrate the procedure involved in determining the truss constants. The truss is a four-panel flat Pratt with panel length l and overall length L. The cross-sectional area of each of the members is taken as A, an arbitrary value. This truss is identical to the one used by Spagnuolo[2]. Using figure 6(a), member stresses are computed separately for the vertical load and the unknown forces X, Y, and Z, each taken separately, and tabulated as in table A. Because l/AE is a constant in this case it is omitted from the tabulation and then applied to the summations only. The use of fractions is for convenience in this particular case. It should not be implied that this is necessary. The location of the "neutral point" and the truss constants may be found by application of the procedures outlined in sections IX-B and X-A. However, here, direct use of the developed equations is resorted to, substituting numerical values from table A.

TABLE A

TRUSS STRESS SUMMATIONS - FOUR-PANEL PRATT TRUSS

Member	$\frac{L}{A} \cdot \frac{A}{l}$	S_P	$S_{X=1}$	$S_{Y=1}$	$S_{Z=1}$	$S_P S_X$	$S_P S_Y$	$S_P S_Z$	S_Y^2	S_X^2	S_Z^2	$S_X S_Z$	$S_X S_Y$	$S_Y S_Z$
KB	1	$-\frac{3}{2}$	1	$\frac{1}{4}$	$-\frac{1}{4}$	$-\frac{3}{2}$	$-\frac{3}{8}$	$\frac{3}{8}$	$\frac{1}{16}$	1	$\frac{1}{16}$	$-\frac{1}{4}$	$\frac{1}{4}$	$-\frac{1}{16}$
KD	1	-2	1	$\frac{1}{2}$	$-\frac{1}{2}$	-2	-1	1	$\frac{1}{4}$	1	$\frac{1}{4}$	$-\frac{1}{2}$	$\frac{1}{2}$	$-\frac{1}{4}$
KE	1	-2	1	$\frac{1}{2}$	$-\frac{1}{2}$	-2	-1	1	$\frac{1}{4}$	1	$\frac{1}{4}$	$-\frac{1}{2}$	$\frac{1}{2}$	$-\frac{1}{4}$
KG	1	$-\frac{3}{2}$	1	$\frac{3}{4}$	$-\frac{3}{4}$	$-\frac{3}{2}$	$-\frac{9}{8}$	$\frac{9}{8}$	$\frac{9}{16}$	1	$\frac{9}{16}$	$-\frac{3}{4}$	$\frac{3}{4}$	$-\frac{9}{16}$
AB	$\sqrt{2}$	$\frac{3}{2}\sqrt{2}$	0	$-\frac{\sqrt{2}}{4}$	$\frac{\sqrt{2}}{4}$	0	$-\frac{3}{4}\sqrt{2}$	$\frac{3}{4}\sqrt{2}$	$\frac{\sqrt{2}}{8}$	0	$\frac{\sqrt{2}}{8}$	0	0	$-\frac{\sqrt{2}}{8}$
BC	1	$-\frac{3}{2}$	0	$\frac{1}{4}$	$-\frac{1}{4}$	0	$-\frac{3}{8}$	$\frac{3}{8}$	$\frac{1}{16}$	0	$\frac{1}{16}$	0	0	$-\frac{1}{16}$
CD	$\sqrt{2}$	$\frac{\sqrt{2}}{2}$	0	$-\frac{\sqrt{2}}{4}$	$\frac{\sqrt{2}}{4}$	0	$-\frac{\sqrt{2}}{4}$	$\frac{\sqrt{2}}{4}$	$\frac{\sqrt{2}}{8}$	0	$\frac{\sqrt{2}}{8}$	0	0	$-\frac{\sqrt{2}}{8}$
DE	1	1	0	0	0	0	0	0	0	0	0	0	0	0
EF	$\sqrt{2}$	$\frac{\sqrt{2}}{2}$	0	$\frac{\sqrt{2}}{4}$	$-\frac{\sqrt{2}}{4}$	0	$\frac{\sqrt{2}}{4}$	$-\frac{\sqrt{2}}{4}$	$\frac{\sqrt{2}}{8}$	0	$\frac{\sqrt{2}}{8}$	0	0	$-\frac{\sqrt{2}}{8}$
FG	1	$-\frac{3}{2}$	0	$-\frac{1}{4}$	$\frac{1}{4}$	0	$\frac{3}{8}$	$-\frac{3}{8}$	$\frac{1}{16}$	0	$\frac{1}{16}$	0	0	$-\frac{1}{16}$
GH	$\sqrt{2}$	$\frac{3}{2}\sqrt{2}$	0	$\frac{\sqrt{2}}{4}$	$-\frac{\sqrt{2}}{4}$	0	$\frac{3}{4}\sqrt{2}$	$-\frac{3}{4}\sqrt{2}$	$\frac{\sqrt{2}}{8}$	0	$\frac{\sqrt{2}}{8}$	0	0	$-\frac{\sqrt{2}}{8}$
AJ	1	0	0	1	0	0	0	0	1	0	0	0	0	0
CJ	1	$\frac{3}{2}$	0	$\frac{3}{4}$	$\frac{1}{4}$	0	$\frac{9}{8}$	$\frac{3}{8}$	$\frac{9}{16}$	0	$\frac{1}{16}$	0	0	$\frac{3}{16}$
FJ	1	$\frac{3}{2}$	0	$\frac{1}{4}$	$\frac{3}{4}$	0	$\frac{3}{8}$	$\frac{9}{8}$	$\frac{1}{16}$	0	$\frac{9}{16}$	0	0	$\frac{3}{16}$
HJ	1	0	0	0	1	0	0	0	0	0	1	0	0	0
					$\sum$	-7	-2	5	$\frac{46}{16} + \frac{\sqrt{2}}{2}$	4	$\frac{46}{16} + \frac{\sqrt{2}}{2}$	-2	2	$-\left(\frac{14}{16} + \frac{\sqrt{2}}{2}\right)$
					$\frac{l}{AE} \cdot \sum$	$-7\frac{l}{AE}$	$-2\frac{l}{AE}$	$5\frac{l}{AE}$	$\frac{1}{8}(23 + 4\sqrt{2})\frac{l}{AE}$	$4\frac{l}{AE}$	$\frac{1}{8}(23 + 4\sqrt{2})\frac{l}{AE}$	$-2\frac{l}{AE}$	$2\frac{l}{AE}$	$-\frac{1}{8}(7 + 4\sqrt{2})\frac{l}{AE}$

1. Location of the "neutral point"

From equation (1.08)

$$\delta = -\frac{1}{2} = -1 \text{ ft}$$

The negative sign indicates that the location of the "neutral point" is below the upper end-panel point of the truss, in this case half way between the top and bottom chords. In this particular case only, since the chords are of constant area, it is known that the location of the "neutral point" is at the gravity axis of the top and bottom chords.

2. Moment stiffness and carryover factor.

From equation (1.11)

$$M_N \Big|_{\varphi_N = 1} = K_T = 0.5657 \text{ AE}$$

and from equation (1.12)

$$\text{carryover factor} \quad r = 0.1167$$

3. Thrust stiffness.

From equation (1.13)

$$T \Big|_{\Delta_N = 1} = K_t = 0.25 \text{ AE}$$

4. Fixed end moment and fixed end thrust.

From the strain energy expression, equation (1.14), the following equation (1.15) is obtained:

$$\left.\begin{aligned}
\frac{\partial U}{\partial X} &= \sum (S_P S_X + X S_X^2 + Y S_X S_Y + Z S_X S_Z)\frac{L}{AE} = 0 \\
\frac{\partial U}{\partial Y} &= \sum (S_P S_Y + X S_X S_Y + Y S_Y^2 + Z S_Y S_Z)\frac{L}{AE} = 0 \\
\frac{\partial U}{\partial Z} &= \sum (S_P S_Z + X S_X S_Z + Y S_Y S_Z + Z S_Z^2)\frac{L}{AE} = 0
\end{aligned}\right\} \qquad (1.15)$$

The solution of these three equations using numerical values from table A is

$$X = 1.75$$

$$Y = -0.75$$

$$Z = -0.75$$

This force system may be resolved by statics into a moment and a thrust at the "neutral point" as shown in figure 6(b).

$$\frac{1}{2} T + \frac{1}{2} M = 1.75$$

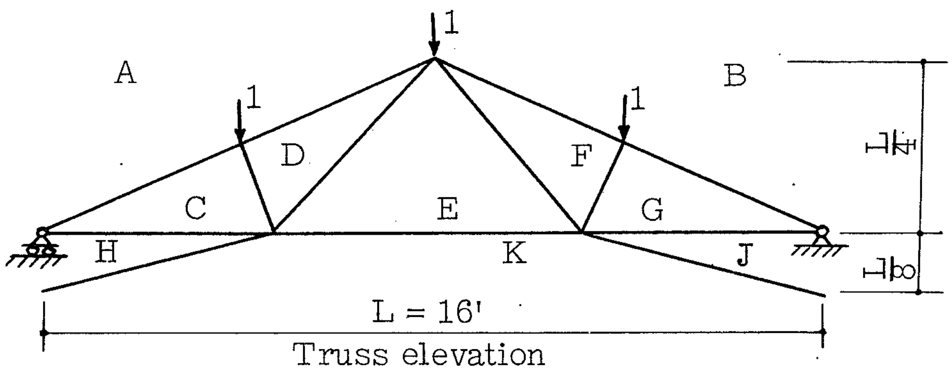
$$-\frac{1}{2} T + \frac{1}{2} M = 0.75$$

$$T = 1.0$$

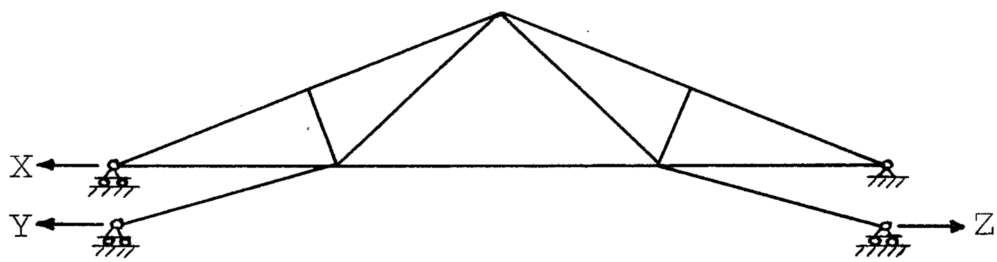
$$M = 2.5$$

C. The Fink truss - a particular case

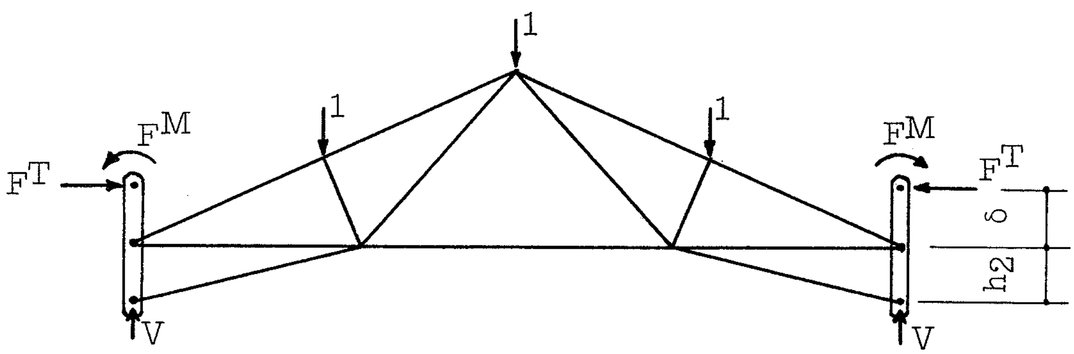
A simplified Fink truss as detailed in figure 7(a) is selected to further illustrate the procedure and to provide numerical values for later use. This particular truss was chosen to correspond to the one used by Vitigliano[4] and Brown[5]. Truss member stresses were computed from figures 7(a) and 7(b) and tabulated in table B. The cross sectional areas of the members are taken as A, an arbitrary value, except for members CD, DE, EF, and FG, which are taken as one half A. The computations in the table are in terms of the truss geometry and a unit panel point loading and the final summations must be multiplied by L/AE and by a load factor P if P is greater than unity. In this particular case L = 16 feet and the second summation line contains the factor 16/AE. The equations developed in section X-A are used directly substituting numerical values from table B.



(a) Load system and truss details.



(b) Redundant forces.



(c) Fixed end moment and thrust.

Figure 7.- Fink truss.

TABLE B.- TRUSS STRESS SUMMATIONS - FINK TRUSS

① Member	② $\frac{L}{A} \cdot \frac{A}{16}$	S_P	$S_{X=1}$	$S_{Y=1}$	$S_{Z=1}$	S_PS_X ②	S_PS_Y ②	S_PS_Z ②	S_X^2 ②	S_Y^2 ②	S_Z^2 ②	S_XS_Y ②	S_XS_Z ②	S_YS_Z ②
AC	0.2795	-3.3541	0	-0.6149	-0.2795	0	0.5765	0.2620	0	0.1057	0.0218	0	0	0.0480
AD	.2795	-2.9069	0	-.6149	-.2795	0	.4996	.2271	0	.1057	.0218	0	0	.0480
BF	.2795	-2.9069	0	-.2795	-.6149	0	.2271	.4996	0	.0218	.1057	0	0	.0480
BG	.2795	-3.3541	0	-.2795	-.6149	0	.2620	.5765	0	.0218	.1057	0	0	.0480
CH	.3125	3.0	1.0	.5500	.25	.9375	.5156	.2344	.3125	.0945	.0195	.1719	.0781	.0430
GJ	.3125	3.0	1.0	1.25	-.45	.9375	1.1719	-.4219	.3125	.4883	.0633	.3906	-.1406	-.1758
EK	.3750	2.0	1.0	1.25	.25	.7500	.9375	.1875	.3750	.5859	.0234	.4688	.0938	.1172
CD	.0699	-0.8944	0	0	0	0	0	0	0	0	0	0	0	0
DE	.1563	1.0	0	.5000	0	0	.0781	0	0	.0391	0	0	0	0
EF	.1563	1.0	0	0	.5000	0	0	.0781	0	0	.0391	0	0	0
FG	.0699	-.8944	0	0	0	0	0	0	0	0	0	0	0	0
HK	.3366	0	0	1.0770	0	0	0	0	0	.3904	0	0	0	0
JK	.3366	0	0	0	1.0770	0	0	0	0	0	.3904	0	0	0
					$\sum$	2.6250	4.2684	1.6434	1.000	1.8533	.7908	1.0313	.0313	.1765
					$\sum \cdot \frac{16}{AE}$	$42.00\frac{1}{AE}$	$68.2944\frac{1}{AE}$	$26.2944\frac{1}{AE}$	$16.0\frac{1}{AE}$	$29.6528\frac{1}{AE}$	$12.6528\frac{1}{AE}$	$16.5008\frac{1}{AE}$	$0.5008\frac{1}{AE}$	$2.8240\frac{1}{AE}$

1. Location of the "neutral point."

From equation (1.08)

$$\delta = 0.0668$$

The positive sign indicates that the location of the "neutral point" is above the upper end-panel point of the truss.

2. Moment stiffness and carryover factor.

From equations (1.11) and (1.12)

$$M_N \Big|_{\varphi_N=1} = K_T = 0.3272 \text{ AE}$$

carryover factor $r = 0.1770$

3. Thrust stiffness.

From equation (1.13)

$$T \Big|_{\Delta_N=1} = K_t = 0.0626 \text{ AE}$$

4. Fixed end moment and fixed end thrust.

Similar to the procedure in section X-B, 4, the equations (1.15) are solved simultaneously to obtain X, Y, and Z as shown in figure 7(b).

$$X = -0.8491$$

$$Y = -1.6714$$

$$Z = -1.6714$$

This force system is resolved by statics, using figure 2(c), into a moment and a thrust acting at point N as shown in figure 7(c).

$$\begin{cases} \dfrac{(\delta + h_2)}{h_2} T + \dfrac{1}{h_2} M = -0.8492 \\ \dfrac{\delta}{h_2} T + \dfrac{1}{h_2} M = +1.6714 \end{cases}$$

$$\begin{cases} 2.0668\ T + M = -1.6984 \\ 0.0668\ T + M = +3.5428 \end{cases}$$

$$T = -2.5205 = pT$$

$$M = 3.5112 = pM$$

The negative sign for T indicates that it acts opposite to the direction assumed in figure 2(c).

XI. THE SUBSTITUTE COLUMN CONSTANTS

A. General

The ordinary definition of moment stiffness, etc., as given in section VIII is commonly applied to the physical end of a solid beam. However, no special definition is required to define these constants about any point, if the location of the point is defined, the correct force systems transferred, and if it is recognized that any joint rotation or translation occurring in the application of moment distribution must now occur at this point and not at the end of the beam. It is necessary only to follow the ordinary definitions precisely in determining the constants. The latter is particularly important in the carryover factor since its magnitude changes as the location of the point changes and since in general it may be different for each end of the substitute column.

B. The derivations

The derivations are simplified by the direct application of moment distribution as shown in plates I, II, IV, V, and VI. The derivations shown have been made for the location of the "neutral point" above the upper end-panel point of the truss or for a positive δ. The derivations hold also for the location of the "neutral point" below the upper end-panel point of the truss or for a negative δ. It is necessary only to insert the proper sign of δ in the expressions. However, to avoid possible confusion, plate VII gives the required constants for each case for a negative δ.

Plate I illustrates the derivation of column moment stiffness for a hinged column base and a positive δ. Points B and C are respectively the corresponding points of attachment of the upper and lower end-panel points of the truss to the real column. The procedure is to remove all restraints on the column base at point A and to rotate the column bracket and the column about point N through the angle φ. A horizontal force H is applied at point A to force the point back to its original location while holding the bracket in its rotated position and against translation. If no rotation of the column at B is allowed this translation of point A, which is $\varphi(h + \delta)$, produces a fixed end moment of $3EI\ \varphi(h + \delta)/h_1^2$ at B. The normal procedure of relaxing the restraint at B and distributing moments in the column is then followed. The column shear H at point A is now determinate. The moment M_N and the horizontal force H required at point N to hold the bracket in the rotated position may be found by statics. The moment M_N for $\varphi = 1$ radian is K_c, the column moment stiffness.

Plate II illustrates the derivation of column shear stiffness and moment due to shear stiffness for a hinged column base and a positive δ. The procedure is to translate the bracket a distance Δ, holding the bracket against rotation. If no rotation of the column at B is allowed a fixed end moment of $3EI\ \Delta/h_1^2$ is created in the column at B. The remaining steps are similar to those described above for plate I. The shear stiffness is given for identification but is not required in this presentation. The important required constant in this case is K_s, the moment due to shear stiffness.

Plate III gives the final algebraic expressions for column moment stiffness K_c, and column carryover factor r_{cNA}, for a positive δ and a fixed end column base. Only the carryover factor from N to A is given. It should be noted that the carryover factor from A to N is different. The latter factor is not required in this presentation. However, if there were a source of varying moment at the base such that a carryover from A to B would occur then this factor would be required. No examples of continuing structure or flexible base supports are given in this presentation. These cases can be handled by standard procedures.

Plate IV shows the derivation for the expressions given in plate III. The procedure is to remove all restraints from the column base at point A and to rotate the column bracket and column through the angle φ. The bracket is then held against further rotation and against translation and the column itself is held against rotation at point B. The bottom end of the column is then returned to its original location at point A, the end being held against rotation. This translation causes fixed end moments of

$$6EI\,\Delta/L^2 = 6EI\,\varphi(h + \delta)/h_1^2$$

at each end of the column section AB. Although the end of the column has been returned to its original location the end is now in a rotated position (perpendicular to the line cd) relative to the original column center line. The end must be rotated through the angle φ to return it to its original position. In rotating the base through the angle φ the moment at the base is decreased by the amount

$$4EI\,\varphi/h_1$$

The moment at the far end B is also decreased by the amount of carryover from the base of

$$r_{c_{AB}}(4EI\,\varphi/h_1) = \frac{1}{2}\,(4EI\,\varphi/h_1)$$

The respective sums are the adjusted fixed end moments in the column section AB. Note that no fixed end moments have been created in the column section BC in any of these operations.

The normal moment distribution procedure is now performed at point B resulting in the final balanced moments at points A and B, as indicated. The column base shear and moment at point A and the horizontal force and moment at point H may now be obtained by statics. The required carryover factor from H to A is obtained as indicated on plate III.

Plate V shows an alternate derivation for the equations given in plate III. In this derivation the absolute deflections of points B and C are used to obtain fixed end moments in the column sections. The final balanced moments are in different algebraic form than the moments in plate V. However, by algebraic manipulation the respective forms can be shown to be identical.

Plate VI shows the derivation of column shear stiffness and moments due to shear stiffness for a positive δ and a fixed column base. The column bracket is displaced a distance Δ while allowing no rotation of the bracket. The column itself is held against rotation at point B causing fixed end moments of

$$6EI\ \Delta/h_1^2$$

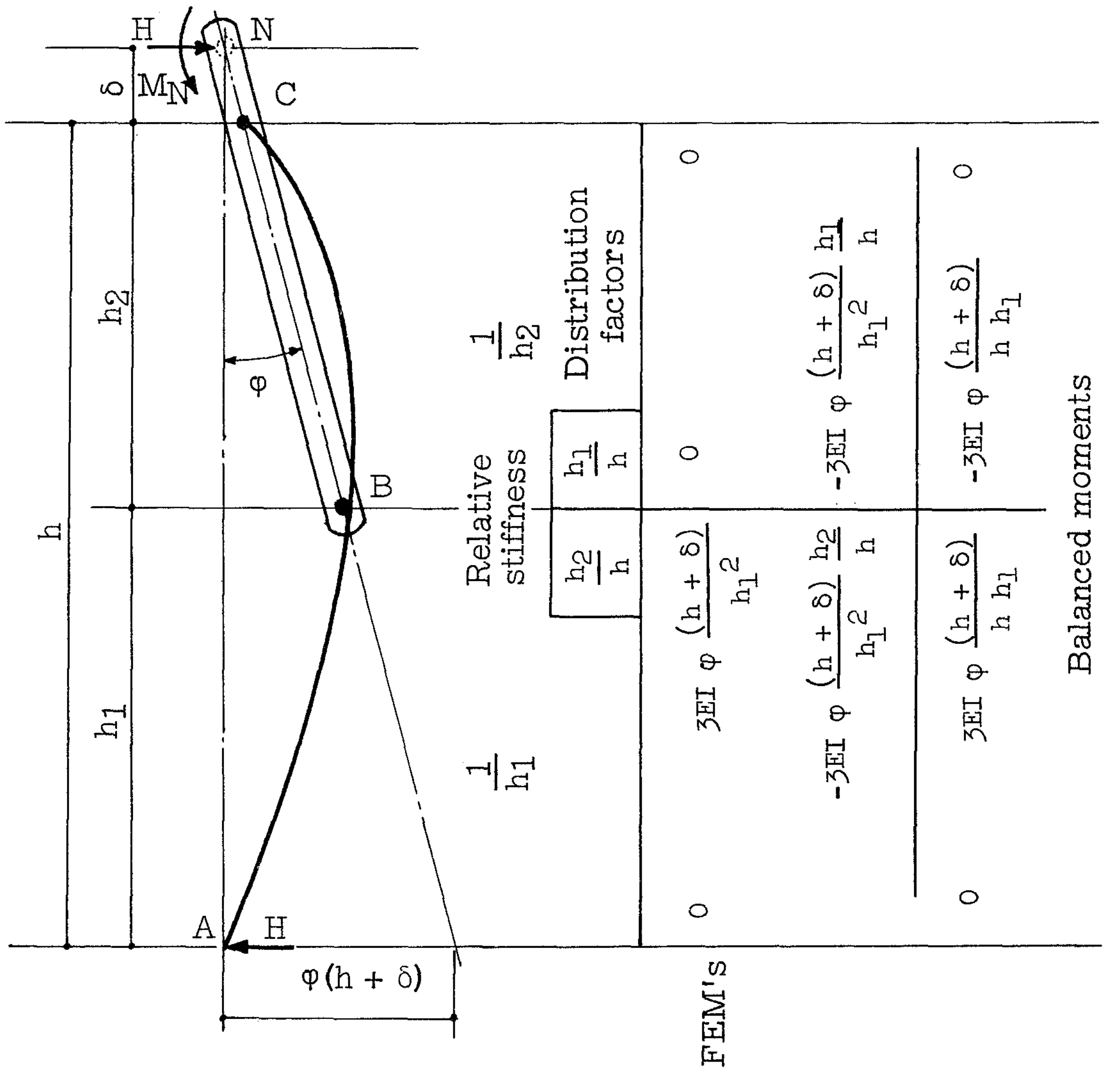
at each end of the column section AB. The moments are then distributed and the required forces and moments found by statics.

Plate VII shows all of the required constants for each of the preceding cases expressed for a negative δ.

PLATE I

COLUMN MOMENT STIFFNESS FOR HINGED COLUMN

BASE AND POSITIVE δ



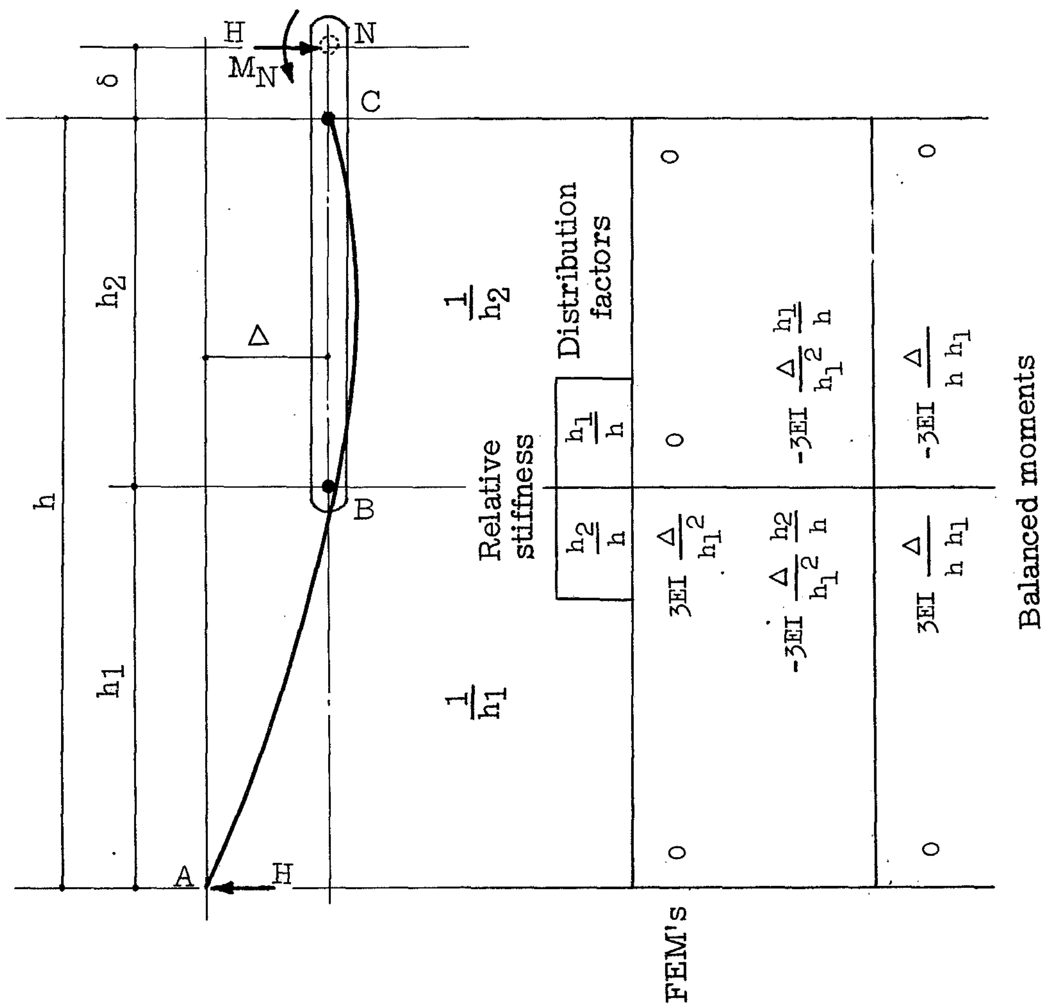
$$H = 3EI\ \varphi \cdot \frac{(h + \delta)}{h\ h_1^{\,2}}$$

$$\text{Moment stiffness}\quad M_N\Big|_{\varphi=1} = K_C = 3EI\ \frac{(h + \delta)^2}{h\ h_1^{\,2}}$$

PLATE II

COLUMN SHEAR STIFFNESS FOR HINGED COLUMN

BASE AND POSITIVE δ



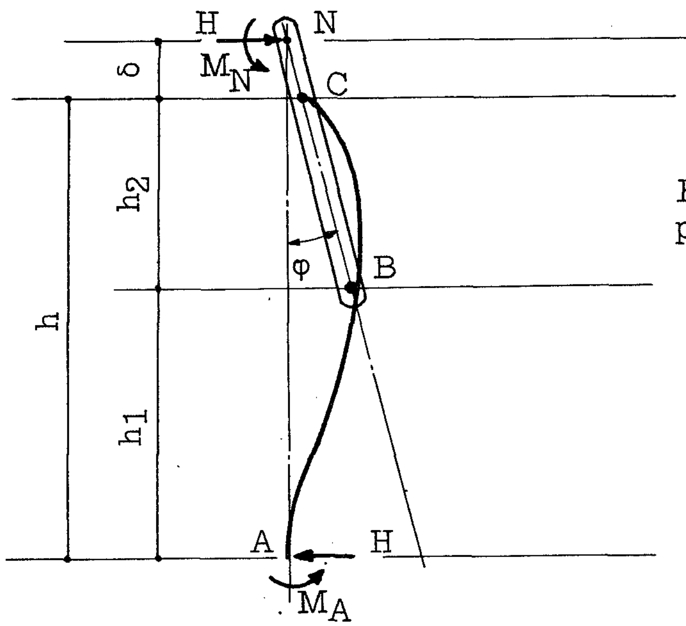
$$H = 3EI\,\frac{\Delta}{h\,h_1^2}$$

Shear stiffness $H\Big|_{\Delta=1} = 3EI\,\frac{1}{h\,h_1^2}$

Moment due to shear stiffness $M_N\Big|_{\Delta=1} = K_S = 3EI\,\frac{(h+\delta)}{h\,h_1^2}$

PLATE III

COLUMN MOMENT STIFFNESS FOR FIXED COLUMN BASE AND POSITIVE δ



For derivations see plates IV and V

$$H_A = 6EI\,\frac{\varphi}{h_1^3}\left[\frac{(h+\delta)(h+2h_1)2 - h_1(2h+h_1)}{4h_2+3h_1}\right]$$

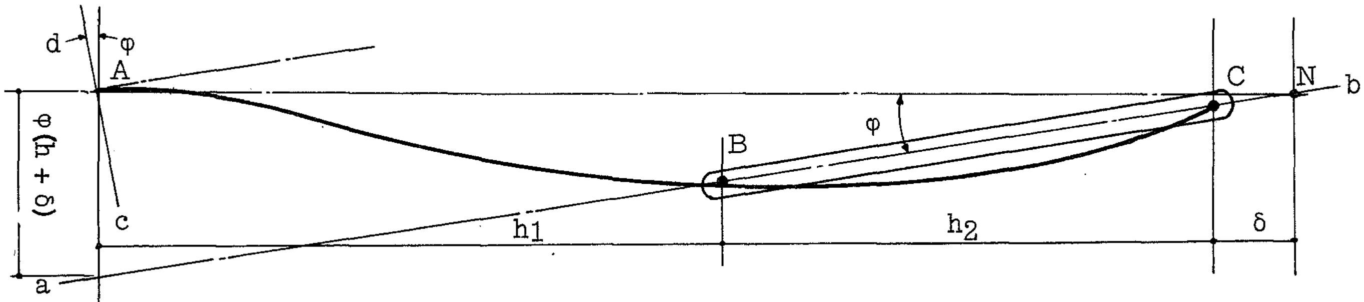
$$M_N\Big|_{\varphi=1} = K_C = 6EI\,\frac{2}{h_1^3}\left[\frac{(h+\delta)^2(h+2h_1) - h_1(h+\delta)(2h+h_1) + h\,h_1^2}{4h_2+3h_1}\right]$$

$$M_A\Big|_{\varphi=1} = 6EI\,\frac{1}{h_1^2}\left[\frac{(h+\delta)(2h+h_1) - 2h\,h_1}{4h_2+3h_1}\right]$$

$$r_{C_{NA}} = \frac{M_A}{M_N}\Bigg|_{\varphi=1} = \frac{1}{2}\,\frac{h_1\left[(h+\delta)(2h+h_1) - 2h\,h_1\right]}{\left[(h+\delta)^2(h+2h_1) - h_1(h+\delta)(2h+h_1) + h\,h_1^2\right]}$$

PLATE IV

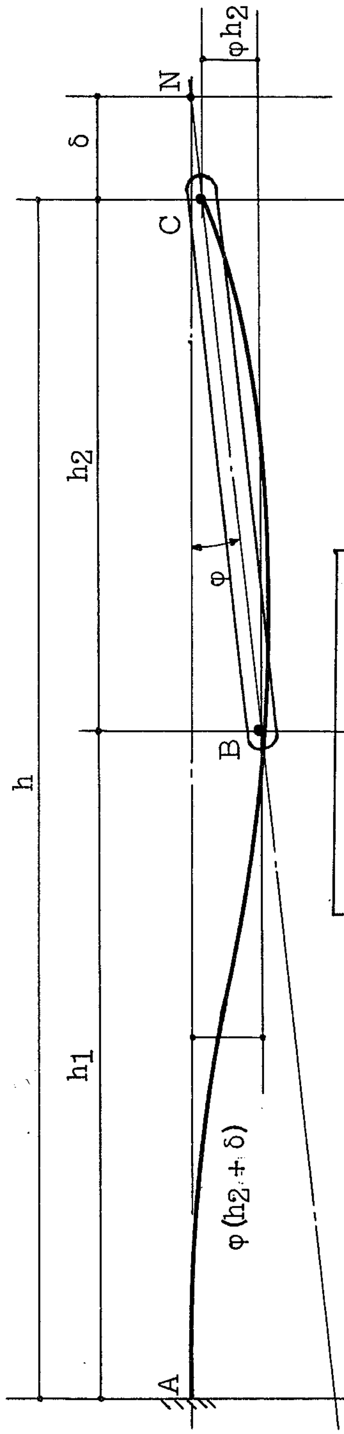
DERIVATION OF COLUMN MOMENTS FOR PLATE III



	A	B (member AB)	B (member BC)	C	
	$\frac{1}{h_1}$		$\frac{3}{4}\,\frac{1}{h_2}$		Relative stiffness
		$\frac{4h_2}{4h_2 + 3h_1}$	$\frac{3h_1}{4h_2 + 3h_1}$		Distribution factors
	$6EI\,\varphi\,\frac{(h+\delta)}{h_1{}^2}$	$6EI\,\varphi\,\frac{(h+\delta)}{h_1{}^2}$	0	0	FEM's
	$-4EI\,\frac{\varphi}{h_1}$ $\times\frac{1}{2}$	$-2EI\,\varphi/h_1$			Balance for end rotation at A of φ
	$-\frac{EI\varphi}{h_1{}^2}\left[\frac{6(h+\delta) - 2h_1}{4h_2 + 3h_1}\right]2h_2$	$-\frac{EI\varphi}{h_1{}^2}\left[6(h+\delta) - 2h_1\right]\frac{4h_2}{4h_2 + 3h_1}$ $\times\frac{1}{2}$	$-\frac{EI\varphi}{h_1{}^2}\left[6(h+\delta) - 2h_1\right]\frac{3h_1}{4h_2 + 3h_1}$		Balance at B
	$6EI\,\frac{\varphi}{h_1{}^2}\left[\frac{(h+\delta)(2h+h_1) - 2h\,h_1}{4h_2 + 3h_1}\right]$	$6EI\,\frac{\varphi}{h_1}\left[\frac{3(h+\delta) - h_1}{4h_2 + 3h_1}\right]$	$-6EI\,\frac{\varphi}{h_1}\left[\frac{3(h+\delta) - h_1}{4h_2 + 3h_1}\right]$	0	Final balanced moments

PLATE V

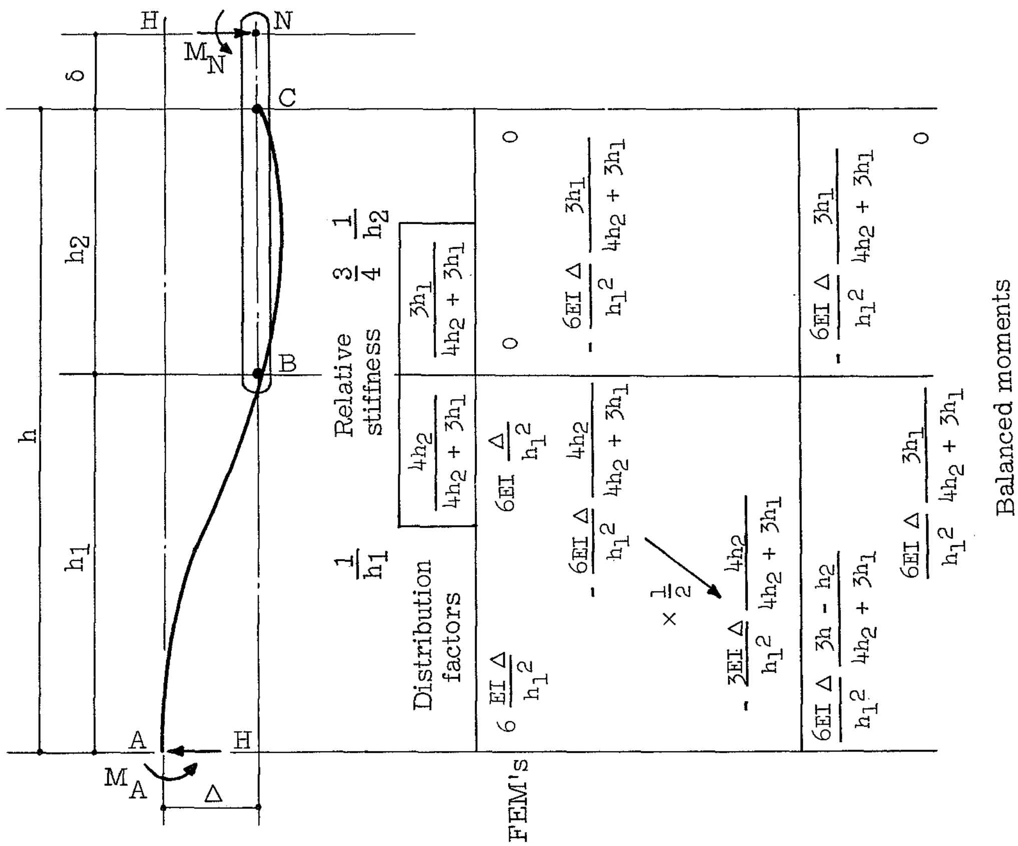
ALTERNATE DERIVATION OF COLUMN MOMENTS FOR PLATE III



	A	B (column BA)	B (column BC)	C
Distribution factors		$\frac{4h_2}{4h_2 + 3h_1}$	$\frac{3h_1}{4h_2 + 3h_1}$	
FEM's	$6EI\,\varphi\,\frac{(h_2 + \delta)}{h_1^2}$	$6EI\,\varphi\,\frac{(h_2 + \delta)}{h_1^2}$	$-3EI\,\varphi\,\frac{h_2}{h_2^2}$	0
		$-3EI\,\varphi\left[\frac{2h_2(h_2 + \delta) - h_1^2}{h_1^2\,h_2}\right]\frac{4h_2}{4h_2 + 3h_1}$	$-3EI\,\varphi\left[\frac{2h_2(h_2 + \delta) - h_1^2}{h_1^2\,h_2}\right]\frac{3h_1}{4h_2 + 3h_1}$	
	$\times\frac{1}{2}$ $-6EI\,\varphi\left[\frac{2h_2(h_2 + \delta) - h_1^2}{h_1^2\,h_2}\right]\frac{h_2}{4h_2 + 3h_1}$			
Final balanced moments	$\frac{6EI\,\varphi}{h_1^2}\left[\frac{(h_2 + \delta)(2h + h_1) + h_1^2}{4h_2 + 3h_1}\right]$	$\frac{6EI\,\varphi}{h_1}\left[\frac{3(h_2 + \delta) + 2h_1}{4h_2 + 3h_1}\right]$	$-6EI\,\frac{\varphi}{h_1}\left[\frac{3(h_2 + \delta) + 2h_1}{4h_2 + 3h_1}\right]$	0

PLATE VI

COLUMN SHEAR STIFFNESS FOR FIXED COLUMN BASE AND POSITIVE δ



$$H = \frac{6EI\,\Delta}{h_1^2}\,\frac{1}{h_1}\left(\frac{2h + 4h_1}{4h_2 + 3h_1}\right)$$

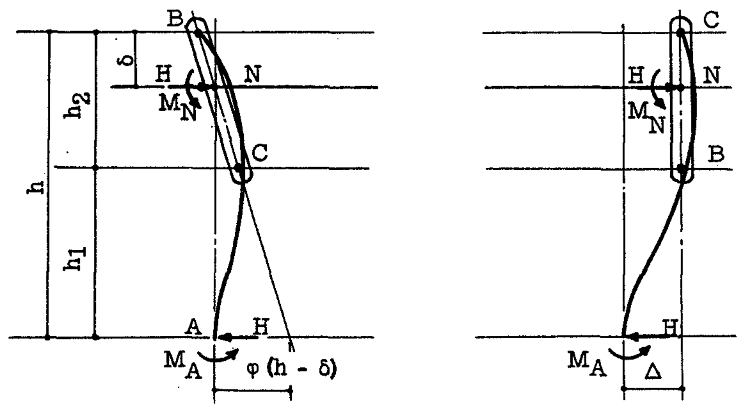
Shear stiffness $\left. H \right|_{\Delta=1} = \frac{6EI}{h_1^3}\left(\frac{2h + 4h_1}{4h_2 + 3h_1}\right)$

Moment due to shear stiffness $\left. M_N \right|_{\Delta=1} = K_S = \frac{6EI}{h_1^2}\left[\frac{(h + \delta)(2h + 4h_1) - h_1(3h - h_2)}{h_1(4h_2 + 3h_1)}\right]$

$$\left. M_A \right|_{\Delta=1} = K_S' = \frac{6EI}{h_1^2}\left(\frac{3h - h_2}{4h_2 + 3h_1}\right)$$

PLATE VII

SUMMARY OF COLUMN CONSTANTS FOR A NEGATIVE δ



1. Hinged column base $M_A = 0$

(a) Moment stiffness $K_C = 3EI \dfrac{(h - \delta)^2}{h\, h_1^2}$

(b) Moment due to shear stiffness $K_S = 3EI \dfrac{(h - \delta)}{h\, h_1^2}$

2. Fixed column base

(a) Moment stiffness

$$K_C = 6EI \frac{2}{h_1^2}\left[\frac{(h - \delta)^2(h + 2h_1) - h_1(h - \delta)(2h + h_1) + h\, h_1^2}{h_1(4h_2 + 3h_1)}\right]$$

(b) Carryover factor

$$r_{CNA} = \frac{1}{2}\, \frac{h_1\left[(h - \delta)(2h + h_1) - 2h\, h_1\right]}{(h - \delta)^2(h + 2h_1) - h_1(h - \delta)(2h + h_1) + h\, h_1^2}$$

(c) Moments due to shear stiffness

$$M_N\Big|_{\Delta=1} = K_S = \frac{6EI}{h_1^2}\left[\frac{(h - \delta)(2h + 4h_1) - h_1(3h - h_2)}{h_1(4h_2 + 3h_1)}\right]$$

$$M_A\Big|_{\Delta=1} = K_S' = \frac{6EI}{h_1^2}\left(\frac{3h - h_2}{4h_2 + 3h_1}\right)$$

XII. THE APPLICATION TO PARALLEL CHORD TRUSSED BENTS

A. Sign convention

If a member tends to rotate the support clockwise, the moment is considered positive.

B. Single bay symmetric bent - four-panel Pratt truss

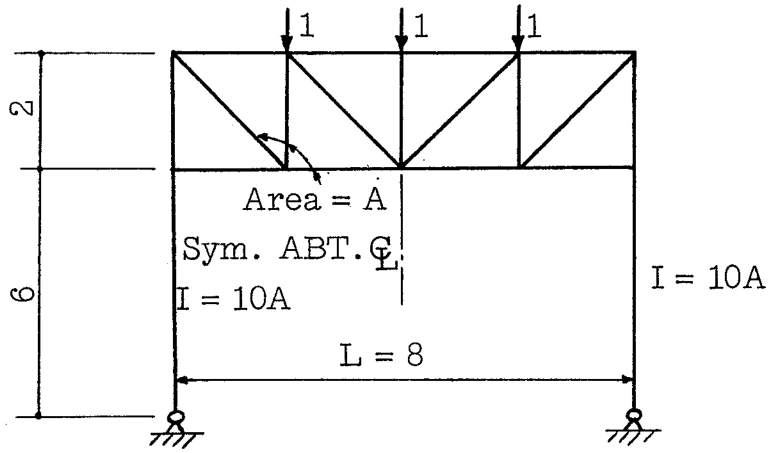
A complete description of the truss is given in figure 6(a) and of the bent in figure 8(a). This purely academic example was chosen for simplicity of the first illustration of method and to provide continuity by use of the truss constants developed in section X-B. The truss constants are

Moment stiffness	$K_T = 0.5657$ AE
Thrust stiffness	$K_t = 0.25$ AE
Carryover factor	$r = 0.1167$
Fixed end moment	$_FM = 2.5$
Fixed end thrust	$_FT = 1.0$
Location of "neutral point"	$\delta = -1.0$ ft

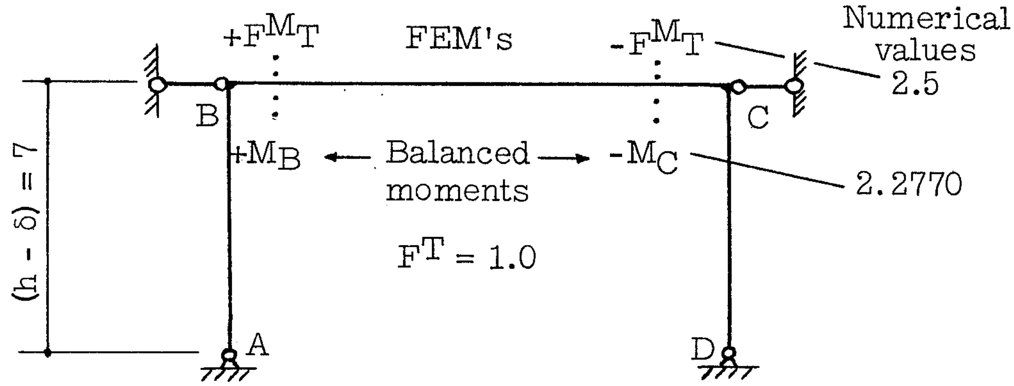
From the equations in plates I and II, using the numerical values from figure 8(a), the column constants are

Moment stiffness	$K_c = 5.1042$ AE
Moment due to shear stiffness	$K_s = 0.7292$ AE

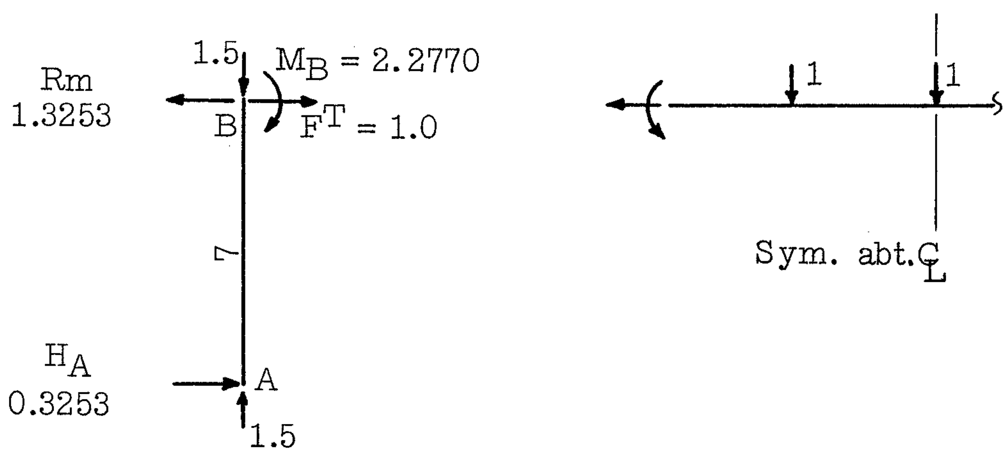
The first step is to attach externally fixed linkages to the "neutral point" as shown in figure 8(b) to prevent translation during the moment balancing process. This is also known as an "auxiliary force system." The translation would occur because of the column shear. The truss FEM's are applied and the moments distributed and balanced according to the usual moment distribution procedure as indicated



(a) Elevation of bent.



(b) Substitute structure and schematic moment distribution for step 1.



(c) Equilibrium diagram for step 1.

Figure 8.- Analysis of single bay symmetric Pratt trussed bent.

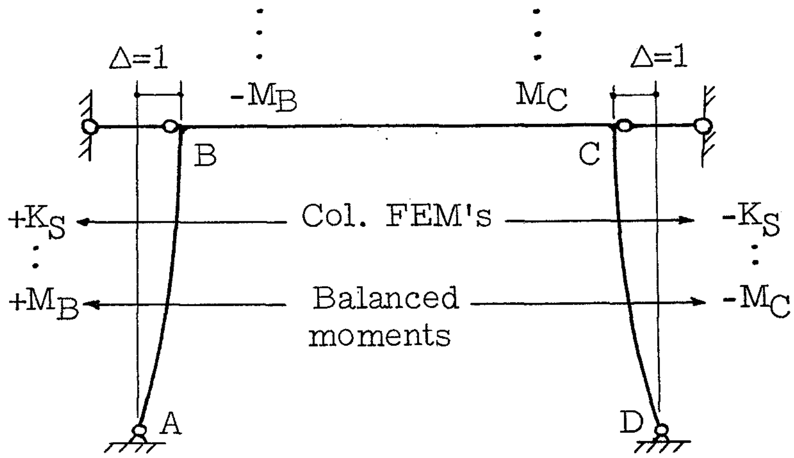
schematically in figure 8(b). Although the process converges rather quickly advantage may be taken of the formulas developed in appendix A. This greatly shortens the numerical calculations and also provides greater accuracy. The balanced moment at the left and right ends of the truss is given by

$$M_B = -M_C = pM \frac{K_C}{K_C + K_T(1 - r)}$$
$$= (2.5)(0.9108)$$
$$= 2.2770$$

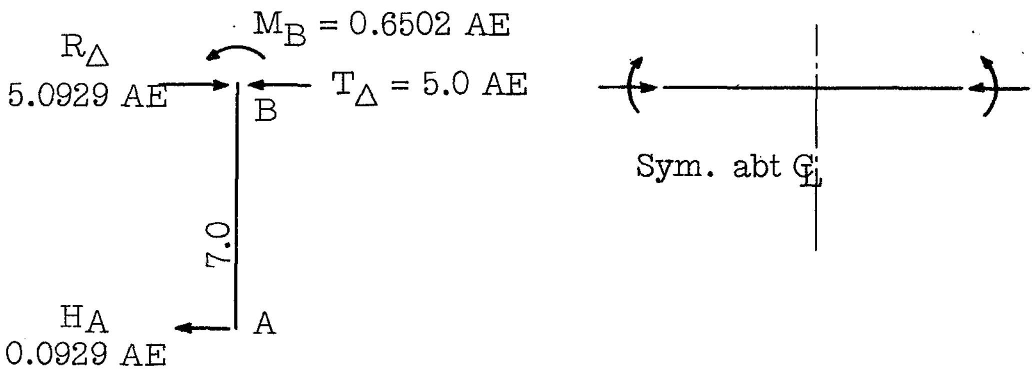
The equilibrium diagram for step 1 is shown in figure 8(c). The fixed linkage provides an additional force R_m at the "neutral point." The fixed end thrust from the truss remains constant throughout the moment balancing process since no translation is allowed. The column base reaction H_A is equal to M_B divided by $(h + s)$.

The second step is to obtain a thrust correction to eliminate the external forces R_m in step 1. Ordinarily the procedure would be to make two correction steps, one for one end of the truss being displaced while the opposite end is held against translation and another for the opposite case. However, in this particular example, because the entire structure is symmetrical about a vertical center line, it is possible to make the thrust correction in one step by displacing both ends of the truss at the same time. Note that the displacements of the ends must both be either toward or away from the symmetrical center line and must both be of the same magnitude.

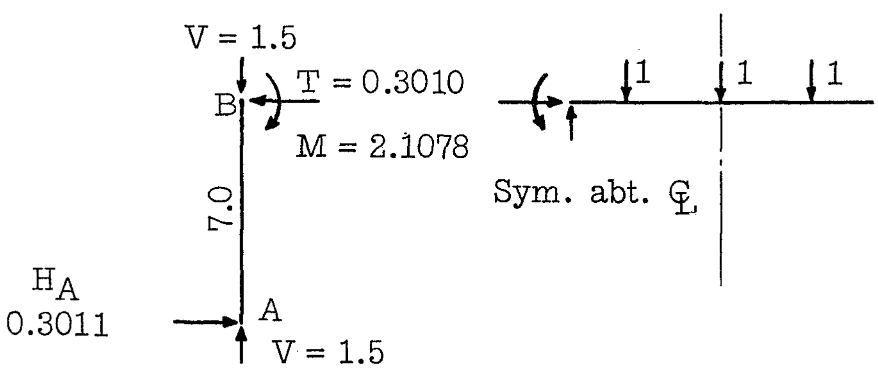
The "neutral point" at each end of the truss is translated a unit Δ toward the center line of the truss, figure 8(d), while preventing the joints from rotating. External fixed linkages are attached to prevent further translation. This translation causes a fixed end moment and a shear



(d) Schematic moment distribution for step 2.



(e) Equilibrium diagram for step 2.



(f) Final equilibrium diagram.

Figure 8.- Concluded.

in the column and a thrust in the truss. The shear in the column is not important since it will vary as the moment is balanced. However, the truss thrust remains constant during the balancing process. Because the ends of the truss have been deflected 2Δ the truss thrust becomes

$$T_\Delta = 2K_t = 2(0.25\ AE) = 0.5\ AE$$

In most of the examples considered in this presentation it has been necessary to make Δ ten units or even one-hundred units in order to obtain sufficient significant figures in the column base shear H. The reason will become evident upon examination of the numerical quantities in the moment calculation to follow. In this particular example a unit Δ is sufficient but $\Delta = 10$ units was used to introduce the idea of larger Δ's. The truss thrust then becomes

$$T_\Delta = 10(0.5\ AE) = 5.0\ AE$$

The column fixed end moment due to the deflection ($10\ K_s$) is applied at the column end and the moments distributed and balanced as indicated schematically in figure 8(d). Again, use is made of the formulas developed in appendix A. The balanced moment at the left (and right) truss end is given by

$$M_B = -M_C = -K_s \frac{K_T(1 - r)}{K_c + K_T(1 - r)}$$

$$= -(0.7292\ AE)(0.08917)10$$

$$= -0.6502\ AE$$

The equilibrium diagram for step 2 is shown in figure 8(e).

To obtain the final equilibrium state the external reactions provided by the fixed linkages must disappear. To accomplish this the following equation must be true.

$$-R_m + a\,R_\Delta = 0$$

$$a = \frac{R_m}{R_\Delta} = \frac{1.3253}{5.0929\ AE} = 0.2602\,\frac{1}{AE}$$

The constant a is the correction factor by which the force system in the equilibrium diagram of step 2, figure 8(e) is multiplied and the product added to the force system in the equilibrium diagram of step 1, figure 8(c). The resultant sum is the final state of equilibrium, figure 8(f). The slight residual (0.0001) in R has been neglected and is not shown in the figure 8(f). This accounts for the difference in T and H. This difference (0.03 percent) is negligible.

For comparison of results the column base shears are compared.

By present method $\quad H_A = 0.3011$

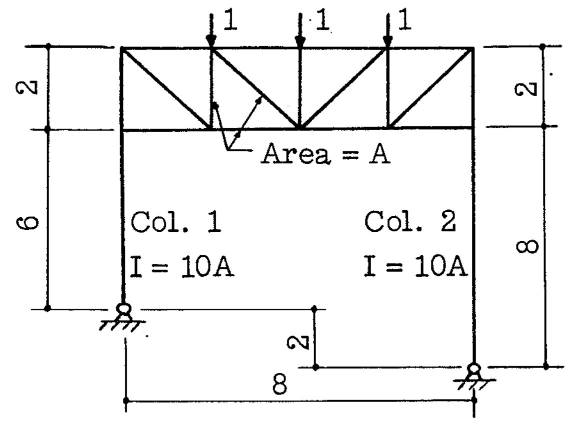
By Least Work method $\quad H_A = 0.30109$

The negligible difference is from calculating errors only. It can be shown that the present method converges to the Least Work solution.

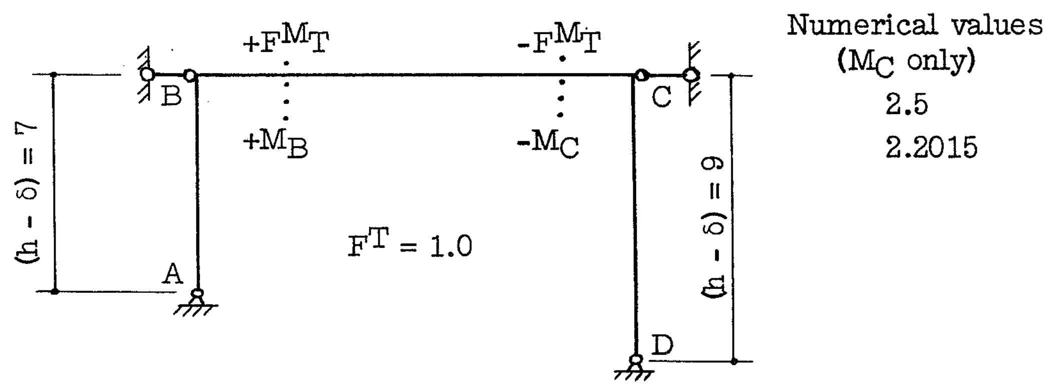
C. Single bay nonsymmetric bent - four-panel Pratt truss

A complete description of the bent is given in figure 9(a). This example was chosen to further illustrate the method and show the steps involved in the analysis of nonsymmetric bents. The numerical work required for the Least Work solution has not been included.

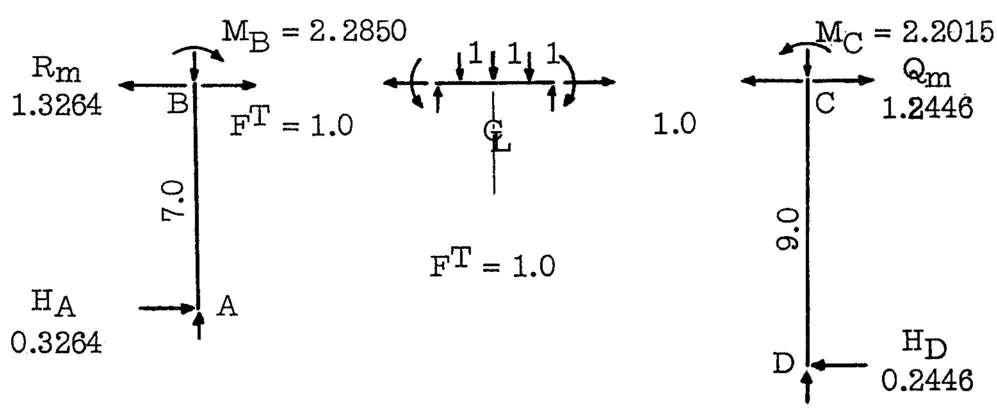
The truss constants are the same as used in the preceding section B. From the equations in plates I and II, using the numerical values from



(a) Elevation of bent.



(b) Substitute structure and schematic moment distribution for step 1.



(c) Equilibrium diagram for step 1.

Figure 9.- Analysis of single bay nonsymmetric Pratt trussed bent.

figure 9(a), the column constants are

Column 1.	Moment stiffness	$K_{c1} = 5.1042$ AE
	Moment due to shear stiffness	$K_{s1} = 0.7292$ AE
Column 2.	Moment stiffness	$K_{c2} = 3.7969$ AE
	Moment due to shear stiffness	$K_{s2} = 0.4219$ AE

Step 1. The first step is to obtain the balanced moments for the truss vertical loading, using an auxiliary force system to prevent translation of the joints. Again the formulas developed in appendix A are used to shorten the numerical calculations. The usual moment distribution method is indicated schematically in figure 9(b). The balanced moment at the left end of the truss is given by

$$\begin{aligned} M_B &= pM_T \frac{K_{c1}(K_{c2} + K_T + rK_T)}{(K_{c1} + K_T)(K_{c2} + K_T) - (rK_T)^2} \\ &= (2.5)(0.9140) \\ &= 2.2850 \end{aligned}$$

The balanced moment at the right end of the truss is given by

$$\begin{aligned} M_C &= -pM_T \frac{K_{c2}(K_{c1} + K_T + rK_T)}{(K_{c1} + K_T)(K_{c2} + K_T) - (rK_T)^2} \\ &= -(2.5)(0.8806) \\ &= -2.2015 \end{aligned}$$

These moments are divided by the respective substitute column lengths to obtain the column shears. The auxiliary restraining forces R_m and Q_m are computed by statics. The equilibrium diagram is shown in figure 9(c).

Step 2. Because the bent is nonsymmetric it is necessary to make two thrust correction calculations - one at each end of the truss. With an auxiliary restraining force at C to prevent translation of C, the left end B is translated an arbitrary distance Δ under an applied force R_{AB}. In this case the translation is made to the right although it may be made in either direction. In this particular example it is convenient to let $\Delta = 100$ units in order to obtain enough significant figures in the value of the column base shear. This means that both the truss thrust stiffness and the moment due to column shear stiffness must be multiplied by the factor 100. The truss thrust becomes

$$T_{AB} = 100\ K_t = 100(0.25\ AE) = 25.0\ AE$$

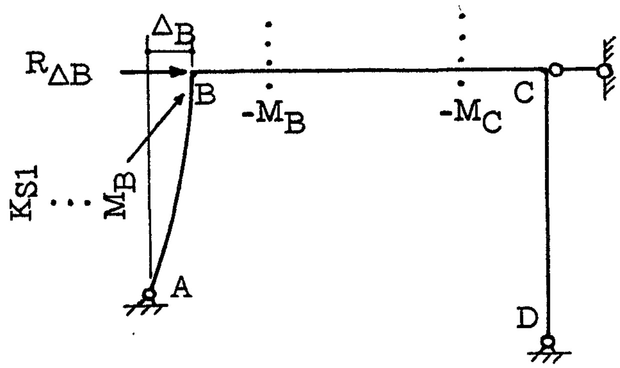
The column fixed end moment is applied at the column end B and the moments distributed and balanced as indicated schematically in figure 9(d). The balanced moment at the left end of the truss is given by

$$\begin{aligned} M_B &= -K_{s1} \frac{K_T(K_{c2} + K_T - r^2 K_T)}{(K_{c1} + K_T)(K_{c2} + K_T) - (rK_T)^2} \\ &= -(0.7292\ AE)(0.099615)\ 100 \\ &= -7.2639\ AE \end{aligned}$$

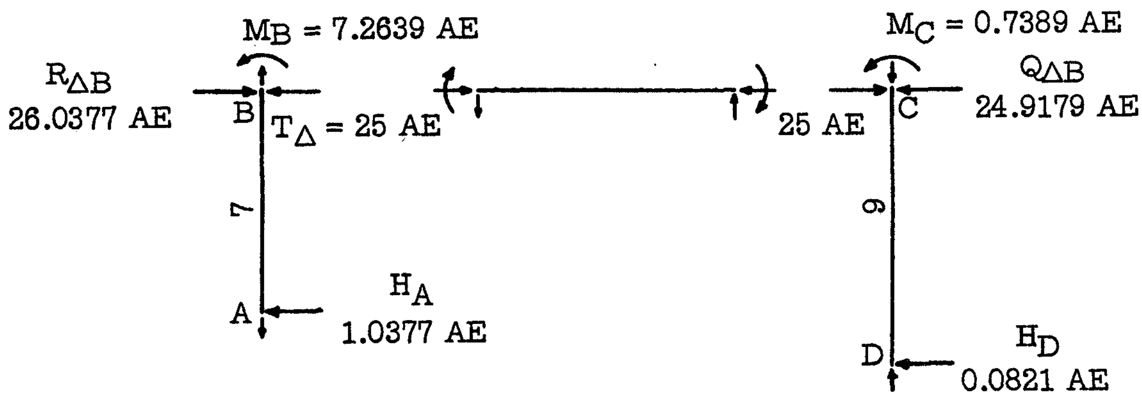
the balanced moment at the right end of the truss is given by

$$\begin{aligned} M_C &= -K_{s1} \frac{K_{c2}\ rK_T}{(K_{c1} + K_T)(K_{c2} + K_T) - (rK_T)^2} \\ &= -(0.7292\ AE)(0.010133)\ 100 \\ &= -0.7389\ AE \end{aligned}$$

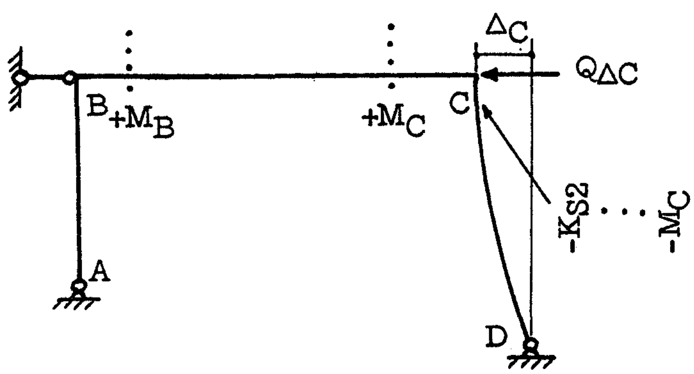
These moments are divided by the respective equivalent column lengths to obtain the column base shears. The impressed force R_{AB} and the auxiliary force Q_{AB} are obtained by statics. The equilibrium diagram is shown in figure 9(e). The vertical forces are not shown because the final values may be computed from the final equilibrium horizontal base reactions.



(d) Schematic moment distribution for step 2.



(e) Equilibrium diagram for step 2.



(f) Schematic moment distribution for step 3.

Figure 9.- Continued.

Step 3. The procedure for thrust correction for a translation of point C while point B is held against translation is similar to step 2. The moment distribution for the column fixed end moments is indicated schematically in figure 9(f). The balanced moment at the left end of the truss is given by

$$M_B = K_{s2} \frac{K_{c1}\, rK_T}{(K_{c1} + K_T)(K_{c2} + K_T) - (rK_T)^2}$$

$$= (0.4219\ AE)(0.013623)\ 100$$

$$= 0.5748\ AE$$

The balanced moment at the right end of the truss is given by

$$M_C = K_{s2} \frac{K_T(K_{c1} + K_T - r^2K_T)}{(K_{c1} + K_T)(K_{c2} + K_T) - (rK_T)^2}$$

$$= (0.4219\ AE)(0.129517)\ 100$$

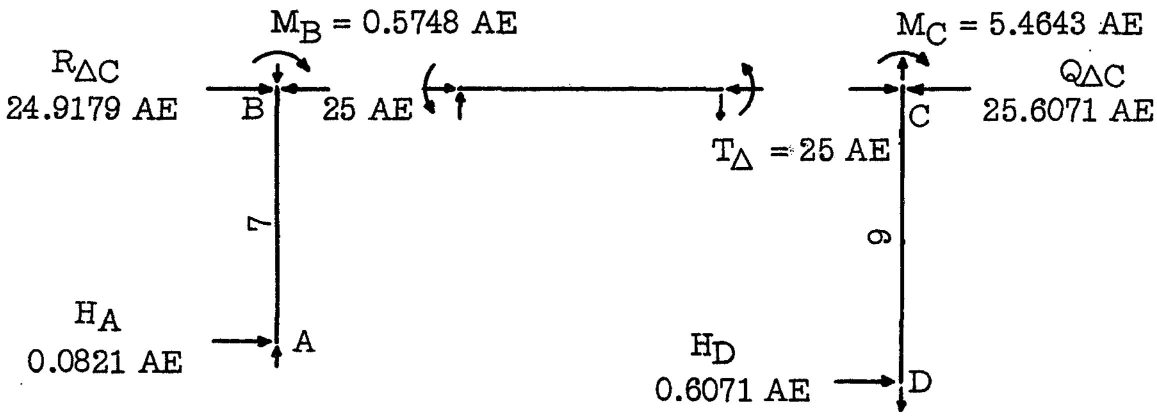
$$= 5.4645\ AE$$

The equilibrium diagram for step 3 is shown in figure 9(g). A check on the numerical calculations in steps 2 and 3 may be made at this point. If the end translation Δ in both steps is the same then it can be shown that the absolute values of H_D obtained in step 2, and H_A obtained in step 3 are identical. See the equilibrium diagrams, figures 9(e) and 9(g).

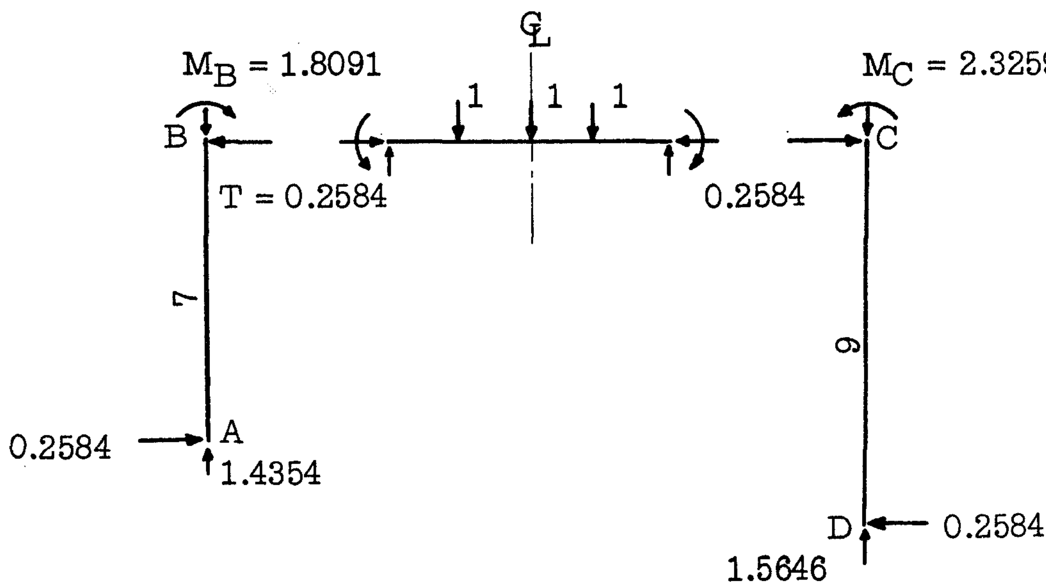
Step 4. To obtain the final equilibrium state the external auxiliary forces must vanish. To accomplish this the following equations must be true.

$$-R_m + aR_{AB} + bR_{AC} = 0$$

$$-Q_m + aQ_{AB} + bQ_{AC} = 0$$



(g) Equilibrium diagram for step 3.



(h) Final equilibrium diagram.

Figure 9.- Concluded.

Expressed in numerical values

$$(26.0377\ AE)a + (24.9179\ AE)b = 1.3264$$

$$(24.9179\ AE)a + (25.6071\ AE)b = 1.2446$$

The solution to these equations is

$$a = 0.0644/AE$$

$$b = -0.01406/AE$$

The constant a is the correction factor by which the force system in the equilibrium diagram of step 2, figure 9(e), is multiplied and the constant b is the correction factor by which the force system in the equilibrium diagram in step 3, figure 9(g) is multiplied. These products are added to the force system in the equilibrium diagram of step 1, figure 9(c). The resultant sum of the respective moments and forces gives the final state of equilibrium, figure 9(h).

The constant b is given to four significant figures in this particular case only to have the final equilibrium diagram, figure 9(h), balance exactly. If b is taken to the first three significant figures there will be some residual (0.4 percent) in the auxiliary forces.

For comparison of results the column base shears are compared

By present method $H = 0.2584$

By Least Work method $H = 0.2584$

It should be noted that the method of thrust correction used automatically takes care of sidesway. For solid member bents a sidesway correction is often made in a similar manner.

D. Single bay nonsymmetric bent with side load

Consider the trussed bent in figure 10(a). This bent is identical to the preceding example except for the lateral forces acting on the column at the levels of the truss chords. This is of course a simplified side load but might be an approximation of a wind load or some other load. The two 1/2-unit forces can be combined into a 1-unit force acting at the "neutral point." A seperate analysis for this loading is not required and in fact steps 1, 2, and 3 of the analysis in the preceding section C can be used if one fact is recognized. The lateral load of 1 must be present in the final equilibrium state. Therefore the correction equations as written in step 4 of section C may be rewritten in the following form:

$$-R_m + aR_{\Delta B} + bR_{\Delta C} = 1$$

$$-Q_m + aQ_{\Delta B} + bQ_{\Delta C} = 0$$

Expressed in numerical values

$$(26.0377\ AE)a + (24.9179\ AE)b = 2.3264$$

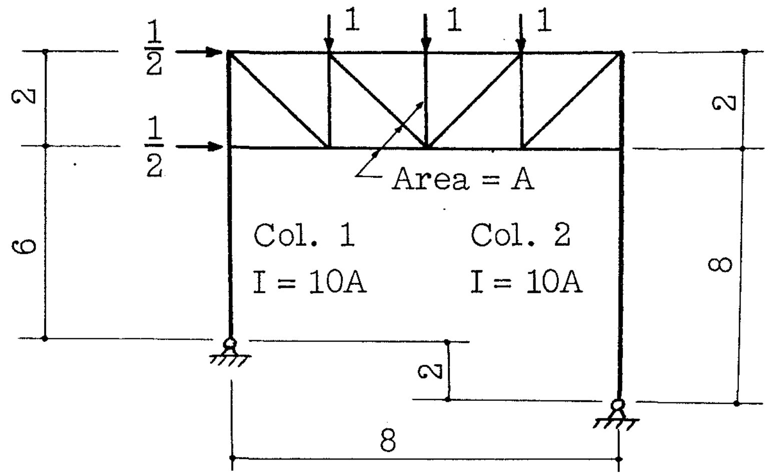
$$(24.9179\ AE)a + (25.6071\ AE)b = 1.2446$$

The solution to these equations is

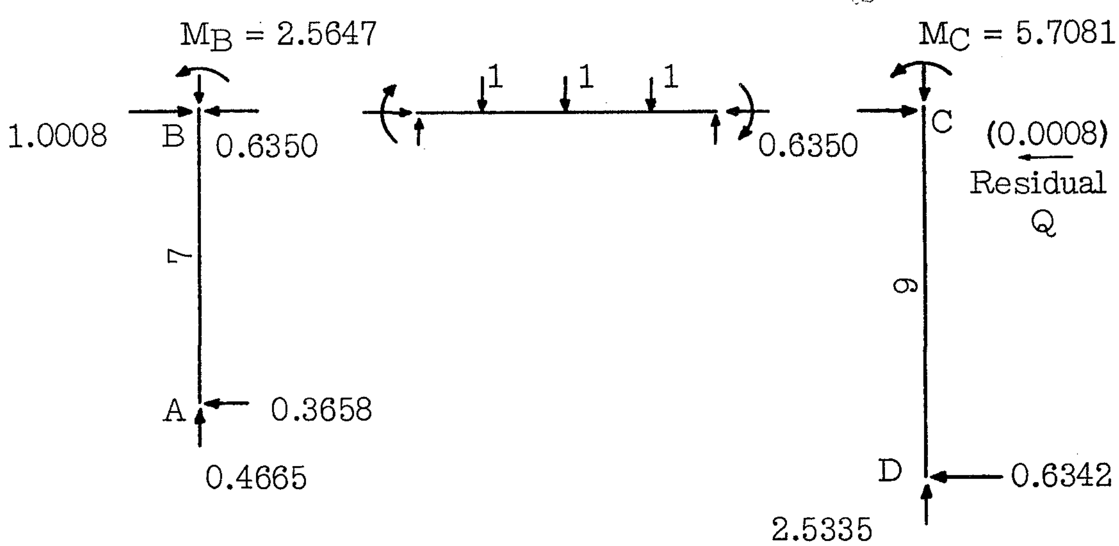
$$a = 0.6229/AE$$

$$b = -0.5575/AE$$

The constant a is the correction factor by which the force system in figure 9(e) is multiplied and the constant b is the correction factor by which the force system in figure 9(g) is multiplied. These products are added to the force system in figure 9(c). The resultant



(a) Elevation of bent.



(b) Final equilibrium diagram.

Figure 10.- Analysis of single bay nonsymmetric bent with side load.

sum of the respective moments and forces gives the final state of equilibrium for the case of the side load, figure 10(b).

Consider the case of a uniform lateral load acting along the entire length of the column. A strict analysis will require the determination of the fixed end moment created in the substitute column and the inclusion of this effect in the analysis of the bent. This case is introduced because it should be noted that a separate step for this load alone cannot be introduced into the thrust correction equations without some manipulation. The simplest way of including extraneous column loadings is to include their fixed end moments in the moment distribution step for the vertical load. The alternate method is of course a separate analysis of the type for step 1 for each type of external loading and to combine the force systems into a single force system for the introduction of the auxiliary forces in the thrust correction equations. This latter approach may be advantageous for considering the effects of various combinations of loading but any saving in work over the former approach is doubtful.

Because lateral wind loading is a common occurrence the required column fixed end moments for a uniform load along the column is given in plate VII. However, it is beyond the scope of this work to include other possible loadings.

E. Single bay symmetric bent - fixed end column bases

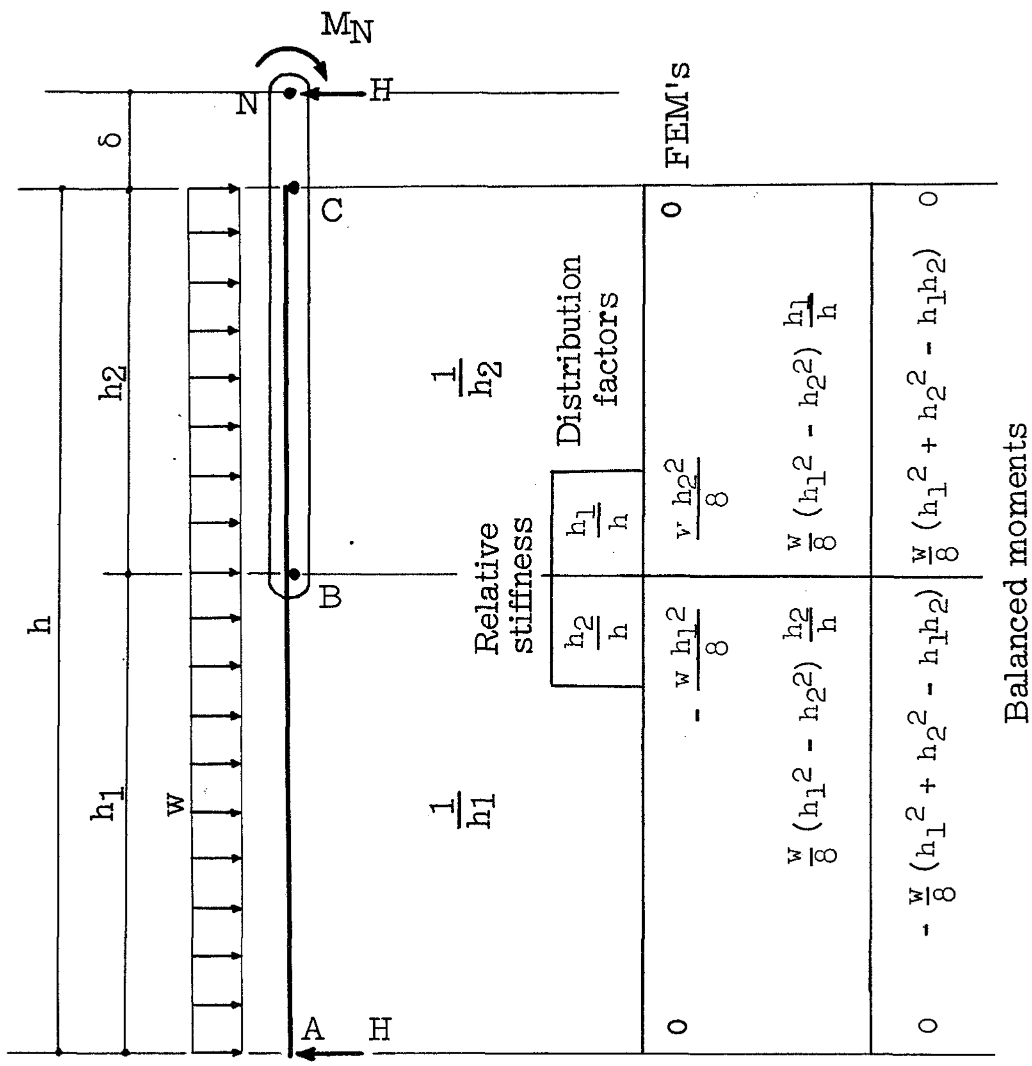
1. <u>Present method of moment distribution.</u>

A complete description of the bent is given in figure 11(a). The truss is an eight-panel Pratt with equal area chords, verticals and

PLATE VIII

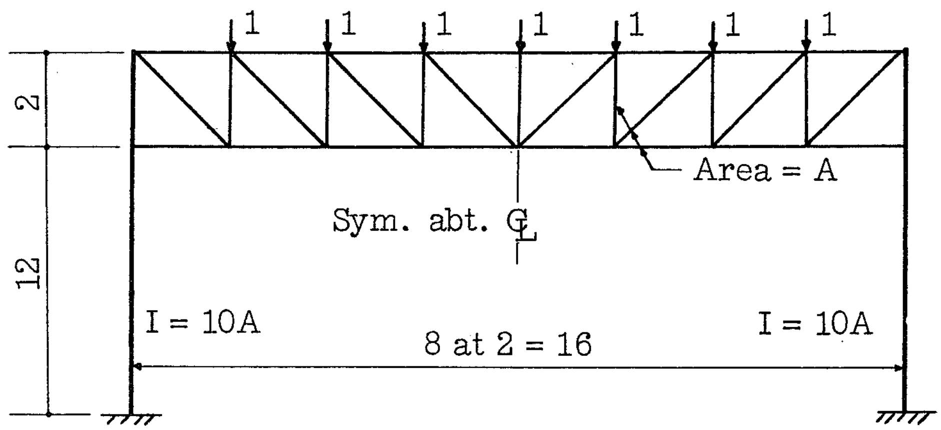
FEM FOR UNIFORM LATERAL LOAD ON COLUMN

FOR HINGED COLUMN BASE AND POSITIVE δ

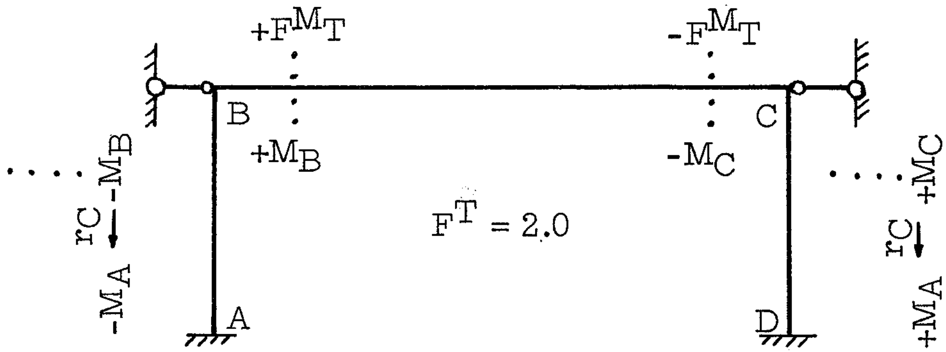


$$H = \frac{w}{8}\left[3h_1^2 - h_2^2 + h_1h_2\right]\frac{1}{h_1}$$

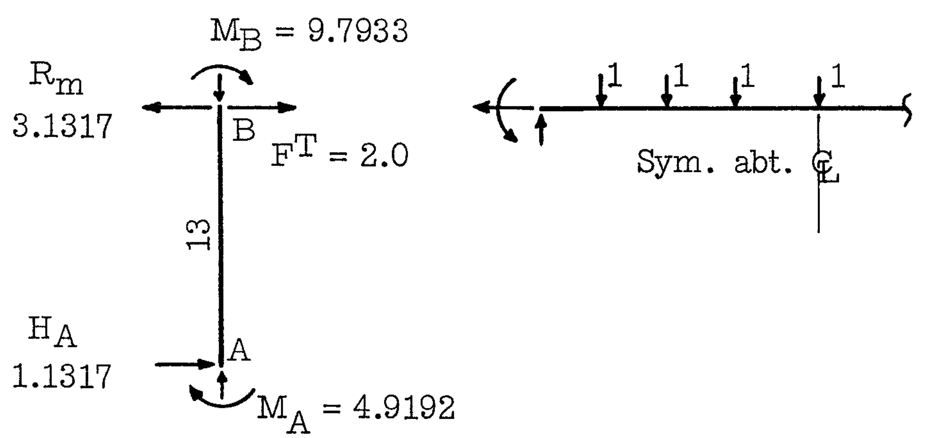
$$M_N = \frac{w}{8}\left[8h\left(\frac{h}{2} + \delta\right) - \frac{(h + \delta)}{h_1}(3h_1^2 - h_2^2 + h_1h_2)\right]$$



(a) Elevation of bent.



(b) Schematic moment distribution for step 1.



(c) Equilibrium diagram for step 1.

Figure 11.- Analysis of single bay symmetric bent with fixed end columns.

diagonals. The Least Work solution for this case is included in abbreviated form at the end of this section. This structure is identical to one used by Rogers[3] (p. 35). It should be noted that his solution by moment and thrust distribution is much closer to the correct Least Work solution than indicated before.

Using the numerical values obtained from the generalized summation formulas developed in appendix B in the equations developed in section X, the truss constants are

Moment stiffness	$K_T = 0.4\ AE$
Thrust stiffness	$K_t = 0.125\ AE$
Carryover factor	$r = 0.3739$
Location of "neutral point"	$\delta = -1$ ft
Fixed end moment	$_F M = 10.5$
Fixed end thrust	$_F T = 2.0$

From the equations in plate VII, using numerical values from figure 11(a) the column constants are

Moment stiffness	$K_c = 3.4601\ AE$
Carryover factor	$r_{C_{MA}} = 0.5023$
Moment due to shear stiffness, top	$K_s = 0.4009\ AE$
Moment due to shear stiffness, bottom	$K_s' = 0.3788\ AE$

Step 1. The first step is to apply the truss fixed end moments and to balance the moments in the structure as indicated schematically in figure 11(b). An auxiliary force system is provided to prevent translation of the joints at B and C. Use is made of the formulas developed

in appendix A to obtain the balanced moments. The balanced moment at each end of the truss becomes

$$M_B = -M_C = \mu M_T \frac{K_c}{K_c + K_T(1 - r)}$$

$$= (10.5)(0.9327)$$

$$= 9.7933$$

The column base moments become

$$M_A = -M_D = -rC_{BA} M_B$$

$$= -(0.5023)(9.7933)$$

$$= -4.9192$$

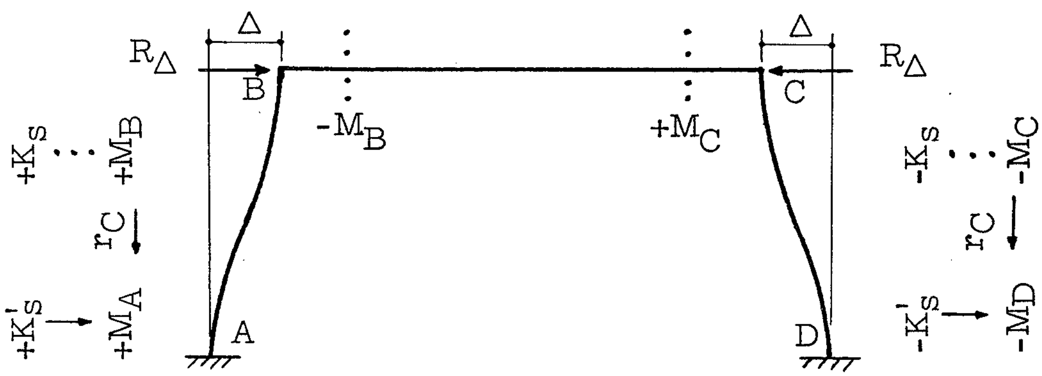
The equilibrium diagram for step 1 is shown in figure 11(c). The column base shears and the auxiliary restraining forces R_m are obtained by statics.

Step 2. The second step is to obtain a thrust correction to eliminate the auxiliary forces obtained in step 1. Again, in this particular example, because the entire structure is symmetrical about a vertical center line, it is possible to make the thrust correction in one step by displacing both ends of the truss at the same time. Note that the displacements of the ends must both be either toward or away from the center line and must be of the same magnitude.

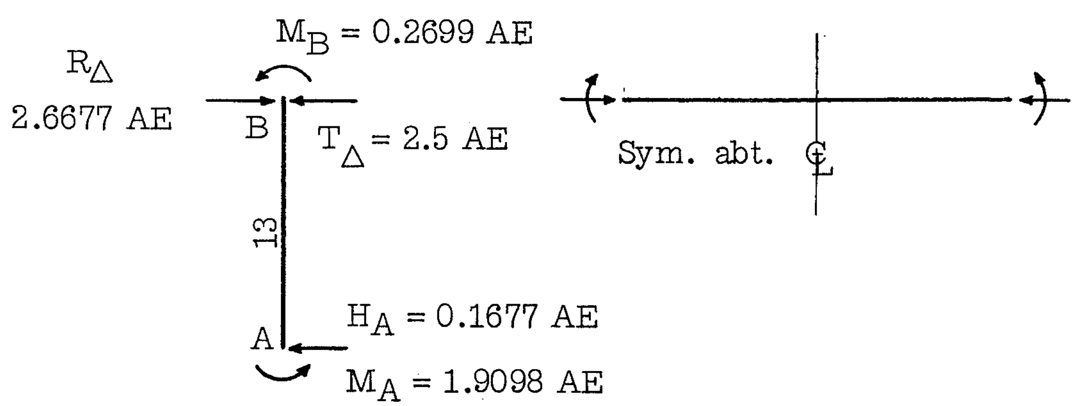
Because the ends of the truss have been displaced 2Δ the truss thrust becomes

$$T_\Delta = 2K_t = 2(0.125\ AE) = 0.25\ AE$$

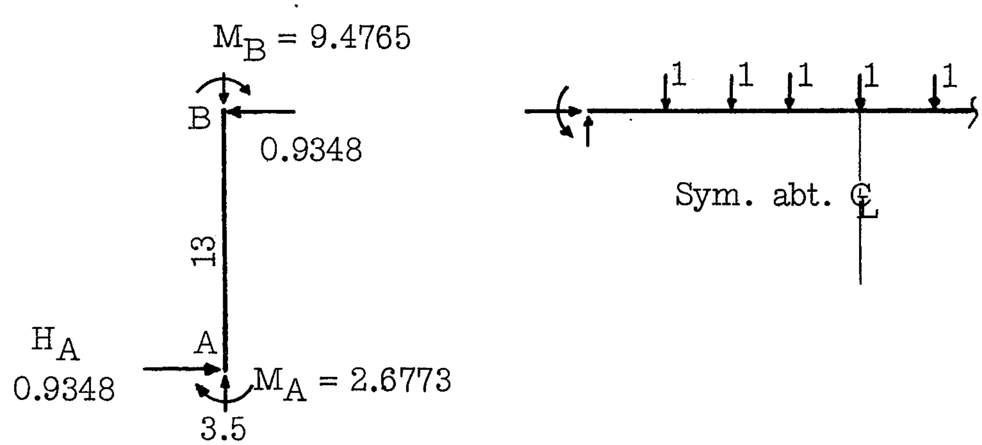
The column fixed end moments are applied at the column ends and the moments distributed and balanced as indicated schematically in figure 11(d).



(d) Schematic moment distribution for step 2.



(e) Equilibrium diagram for step 2.



(f) Final equilibrium diagram.

Figure 11.- Concluded.

Here again it is convenient to make the deflection $\Delta = 10$ units instead of $\Delta = 1$ unit in order to retain enough significant figures in the column shears. The truss thrust now becomes

$$T_\Delta = 10(0.25\ AE) = 2.5\ AE$$

The balanced truss moment becomes

$$M_B = -M_C = -K_s \frac{K_T(1 - r)}{K_c + K_T(1 - r)}$$

$$= -(0.4009\ AE)(0.06732)10$$

$$= -0.2699\ AE$$

The column base moment becomes

$$M_A = -M_D = K_s' - r_{C_{BA}} K_s \frac{K_c}{K_c + K_T(1 - r)}$$

$$= (0.3788\ AE)10 - (0.5023)(0.4009\ AE)(0.9327)10$$

$$= 1.9098\ AE$$

The equilibrium diagram for step 2 is shown in figure 11(e).

Step 3. The equilibrium or thrust correction equation becomes

$$-R_m + aR = 0$$

$$a = \frac{3.1317}{2.6677\ AE} = 1.1739/AE$$

The constant a is the correction factor by which the force system in the equilibrium diagram of step 2, figure 11(e) is multiplied and the product added to the force system in the equilibrium diagram of step 1, figure 11(c). The resultant sum is the final state of equilibrium, figure 11(f).

For comparison of results the column base shear and base moment are compared.

	H_A	M_A
By present method	0.9348	2.6773
By Least Work method	0.9350	2.6774

2. Abbreviated Least Work solution.

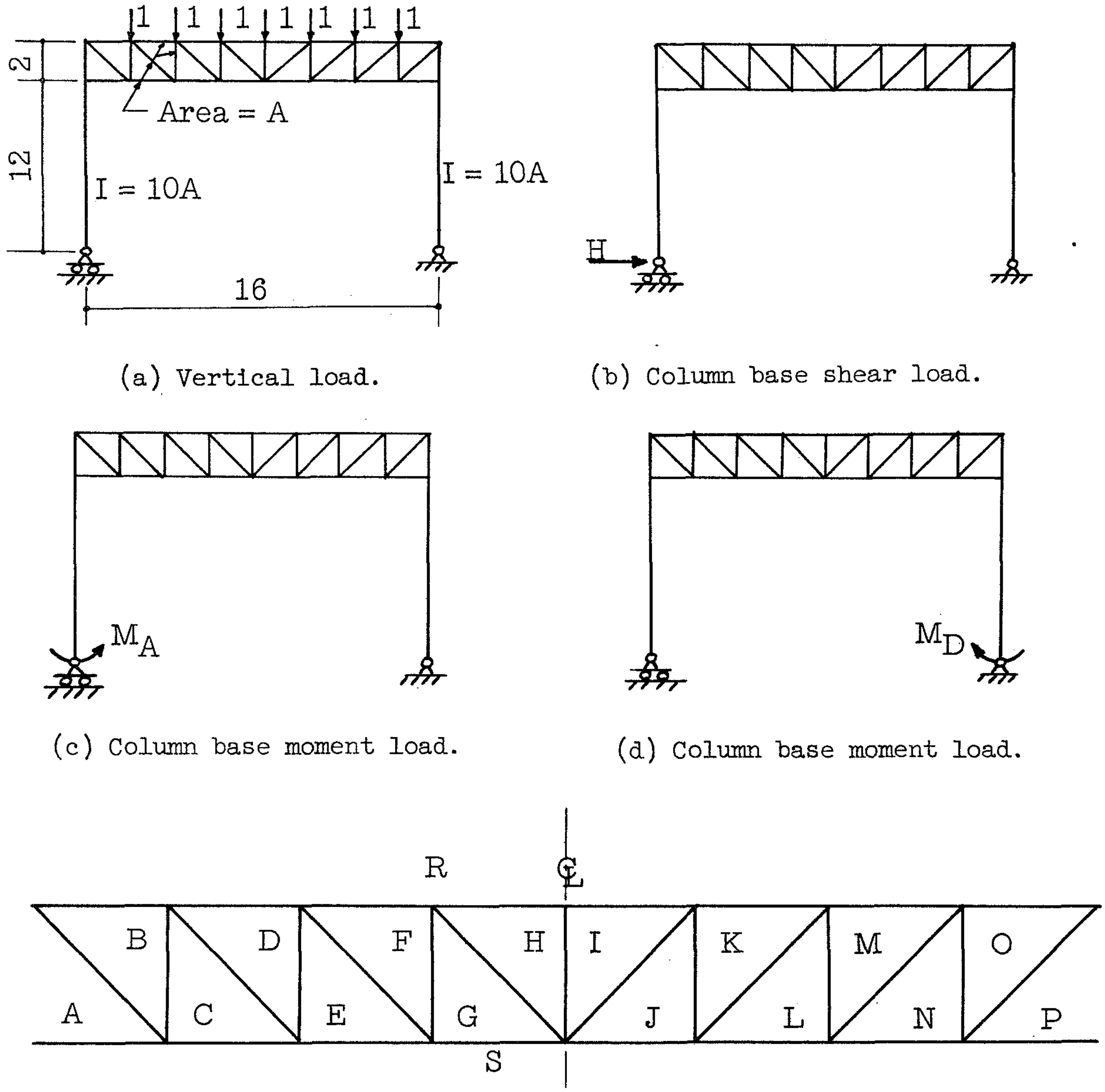
For the Least Work solution of the bent in figure 11(a) the structure is made statically determinate as shown in figure 12(a) and the truss member stresses computed. Next, truss member stresses are computed for each of the cases shown in figures 12(b), 12(c), and 12(d) for a unit force or moment in place of the unknowns. The truss member stresses are tabulated in table C. Figure 12(e) identifies the truss members. For convenience the column L/AE is multiplied by $AE/1$ and the final summations corrected in the second summation line of the table. Also certain columns are multiplied by a numerical factor to eliminate fractions. The subsummations are then divided by the respective factors.

The total strain energy in the structure is expressed by

$$U = \frac{1}{2}\sum\left(S_P + H\,S_H + M_A\,S_{M_A} + M_D\,S_{M_D}\right)^2 \frac{L}{AE} + \frac{1}{2}\int_0^{12} \frac{(H\,y + M_A)^2}{EI}\,dy +$$

$$\frac{1}{2}\int_0^{2} \frac{(6H\,y + \frac{1}{2}M_A\,y)^2}{EI}\,dy + \frac{1}{2}\int_0^{12} \frac{(H\,y + M_D)^2}{EI}\,dy +$$

$$\frac{1}{2}\int_0^{2} \frac{(6H\,y + \frac{1}{2}M_D\,y)^2}{EI}\,dy = 0$$



(a) Vertical load.

(b) Column base shear load.

(c) Column base moment load.

(d) Column base moment load.

(e) Truss member identification.

Figure 12.- Diagrams for Least Work solution of trussed bent.

TABLE C.- TRUSS STRESS SUMMATIONS FOR LEAST WORK SOLUTION

Member	$\frac{L}{A} \cdot \frac{A}{l}$ (2)	$S_{P=1}$	$S_{H=1}$	$S_{M_A=1}$	$S_{M_D=1}$	S_PS_H (2)	$S_PS_{M_A}$ (2) $\times 32$	$S_PS_{M_D}$ (2) $\times 32$	S_H^2 (2)	$S_{M_A}^2$ (2) $\times 256$	$S_{M_D}^2$ (2) $\times 256$	$S_HS_{M_A}$ (2) $\times 16$	$S_HS_{M_D}$ (2) $\times 16$	$S_{M_A}S_{M_D}$ (2) $\times 256$
RA	1	-7/2	6	7/16	1/16	-21	-49	-7	36	49	1	42	6	7.
RD	1	-6	6	6/16	2/16	-36	-72	-24	36	36	4	36	12	12
RF	1	-15/2	6	5/16	3/16	-45	-75	-45	36	25	9	30	18	15
RH	1	-8	6	4/16	4/16	-48	-72	-72	36	16	16	24	24	16
RI	1	-8	6	4/16	4/16	-48	-72	-72	36	16	16	24	24	16
RK	1	-15/2	6	3/16	5/16	-45	-45	-75	36	9	25	18	30	15
RM	1	-6	6	2/16	6/16	-36	-24	-72	36	4	36	12	36	12
RO	1	-7/2	6	1/16	7/16	-21	-7	-49	36	1	49	6	42	7
AB	$\sqrt{2}$	$7\sqrt{2}/2$	0	$\sqrt{2}/16$	$-\sqrt{2}/16$	0	$14\sqrt{2}$	$-14\sqrt{2}$	0	$2\sqrt{2}$	$2\sqrt{2}$	0	0	$-2\sqrt{2}$
CD	$\sqrt{2}$	$5\sqrt{2}/2$	0	$\sqrt{2}/16$	$-\sqrt{2}/16$	0	$10\sqrt{2}$	$-10\sqrt{2}$	0	$2\sqrt{2}$	$2\sqrt{2}$	0	0	$-2\sqrt{2}$
EF	$\sqrt{2}$	$3\sqrt{2}/2$	0	$\sqrt{2}/16$	$-\sqrt{2}/16$	0	$6\sqrt{2}$	$-6\sqrt{2}$	0	$2\sqrt{2}$	$2\sqrt{2}$	0	0	$-2\sqrt{2}$
GH	$\sqrt{2}$	$\sqrt{2}/2$	0	$\sqrt{2}/16$	$-\sqrt{2}/16$	0	$2\sqrt{2}$	$-2\sqrt{2}$	0	$2\sqrt{2}$	$2\sqrt{2}$	0	0	$-2\sqrt{2}$
JI	$\sqrt{2}$	$\sqrt{2}/2$	0	$-\sqrt{2}/16$	$\sqrt{2}/16$	0	$-2\sqrt{2}$	$2\sqrt{2}$	0	$2\sqrt{2}$	$2\sqrt{2}$	0	0	$-2\sqrt{2}$
KL	$\sqrt{2}$	$3\sqrt{2}/2$	0	$-\sqrt{2}/16$	$\sqrt{2}/16$	0	$-6\sqrt{2}$	$6\sqrt{2}$	0	$2\sqrt{2}$	$2\sqrt{2}$	0	0	$-2\sqrt{2}$
MN	$\sqrt{2}$	$5\sqrt{2}/2$	0	$-\sqrt{2}/16$	$\sqrt{2}/16$	0	$-10\sqrt{2}$	$10\sqrt{2}$	0	$2\sqrt{2}$	$2\sqrt{2}$	0	0	$-2\sqrt{2}$
OP	$\sqrt{2}$	$7\sqrt{2}/2$	0	$-\sqrt{2}/16$	$\sqrt{2}/16$	0	$-14\sqrt{2}$	$14\sqrt{2}$	0	$2\sqrt{2}$	$2\sqrt{2}$	0	0	$-2\sqrt{2}$
BC	1	-7/2	0	-1/16	1/16	0	7	-7	0	1	1	0	0	-1
DE	1	-5/2	0	-1/16	1/16	0	5	-5	0	1	1	0	0	-1
FG	1	-3/2	0	-1/16	1/16	0	3	-3	0	1	1	0	0	-1
HI	1	-1	0	0	0	0	0	0	0	1	1	0	0	-1
JK	1	-3/2	0	1/16	-1/16	0	-3	3	0	1	1	0	0	-1
LM	1	-5/2	0	1/16	-1/16	0	-5	5	0	1	1	0	0	-1
NO	1	-7/2	0	1/16	-1/16	0	-7	7	0	1	1	0	0	-1
SA	1	0	-7	-1/2	0	0	0	0	49	64	0	14	0	0
SC	1	7/2	-7	-7/16	-1/16	-49/2	-49	-7	49	49	1	49	7	7
SE	1	12/2	-7	-6/16	-2/16	-42	-72	-24	49	36	4	42	14	12
SG	1	15/2	-7	-5/16	-3/16	-105/2	-75	-45	49	25	9	35	21	15
SJ	1	15/2	-7	-3/16	-5/16	-105/2	-45	-75	49	9	25	21	35	15
SL	1	12/2	-7	-2/16	-6/16	-42	-24	-72	49	4	36	14	42	12
SN	1	7/2	-7	-1/16	-7/16	-49/2	-7	-49	49	1	49	7	49	7
SP	1	0	-7	0	-1/2	0	0	0	49	0	64	0	14	0
Sub $\sum \div$ col. factor						-538	$-\frac{672}{32}$	$-\frac{672}{32}$	680	$\frac{175}{128} + \frac{\sqrt{2}}{16}$	$\frac{175}{128} + \frac{\sqrt{2}}{16}$	26	26	$\frac{81}{128} - \frac{\sqrt{2}}{16}$
$l = 2;\ \therefore\ \frac{2}{AE} \times$ sub $\sum$						$-1076\frac{1}{AE}$	$-42\frac{1}{AE}$	$-42\frac{1}{AE}$	$1360\frac{1}{AE}$	$\left(\frac{175}{64} + \frac{\sqrt{2}}{8}\right)\frac{1}{AE}$	$\left(\frac{175}{64} + \frac{\sqrt{2}}{8}\right)\frac{1}{AE}$	$52\frac{1}{AE}$	$52\frac{1}{AE}$	$\left(\frac{81}{64} - \frac{\sqrt{2}}{8}\right)\frac{1}{AE}$

By the application of Castigliano's second theorem the redundants H, M_A, and M_D may be determined.

$$\frac{\partial U}{\partial H} = 0; \quad \frac{\partial U}{\partial M_A} = 0; \quad \frac{\partial U}{\partial M_D} = 0$$

By performing the integrations and substituting numerical values from table C for the stress summations three equations in the three unknowns are obtained.

$$1494.4\, H + 60\, M_A + 60\, M_D = 1076$$

$$60\, H + \left(\frac{175}{64} + \frac{38}{30} + \frac{\sqrt{2}}{8}\right) M_A + \left(\frac{81}{64} - \frac{\sqrt{2}}{8}\right) M_D = 42$$

$$60\, H + \left(\frac{81}{64} - \frac{\sqrt{2}}{8}\right) M_A + \left(\frac{175}{64} + \frac{38}{30} + \frac{\sqrt{2}}{8}\right) M_D = 42$$

The solution to these equations is

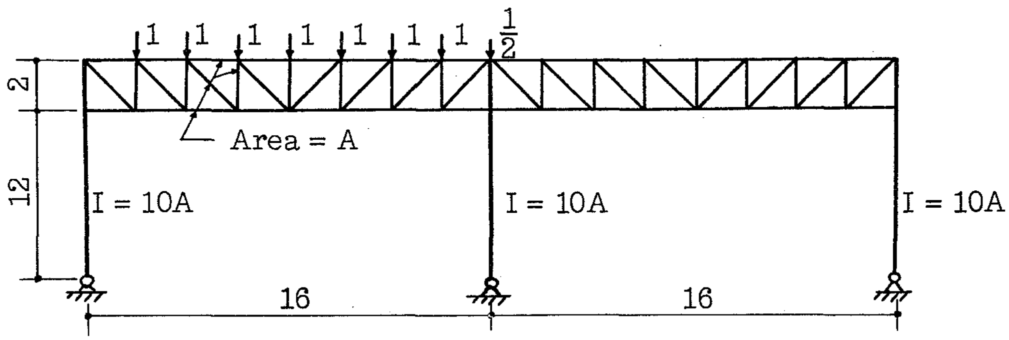
$$H = 0.9350$$

$$M_A = -2.6774$$

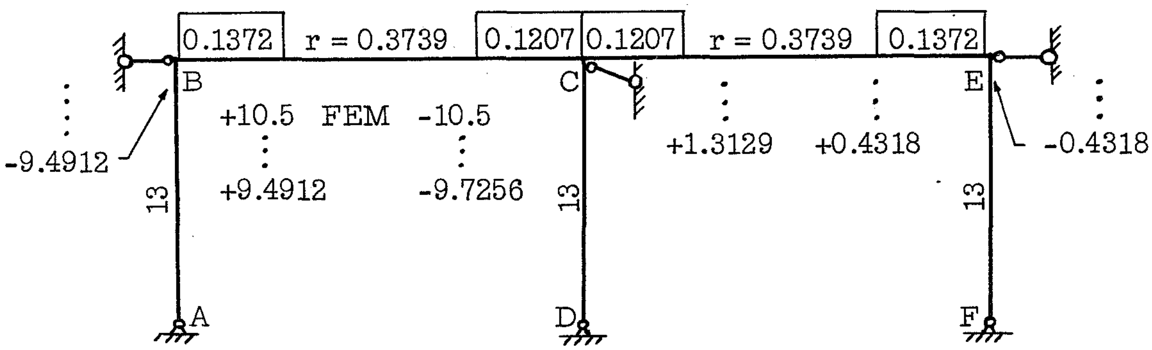
$$M_D = -2.6774$$

F. Two bay structure - nonsymmetric load

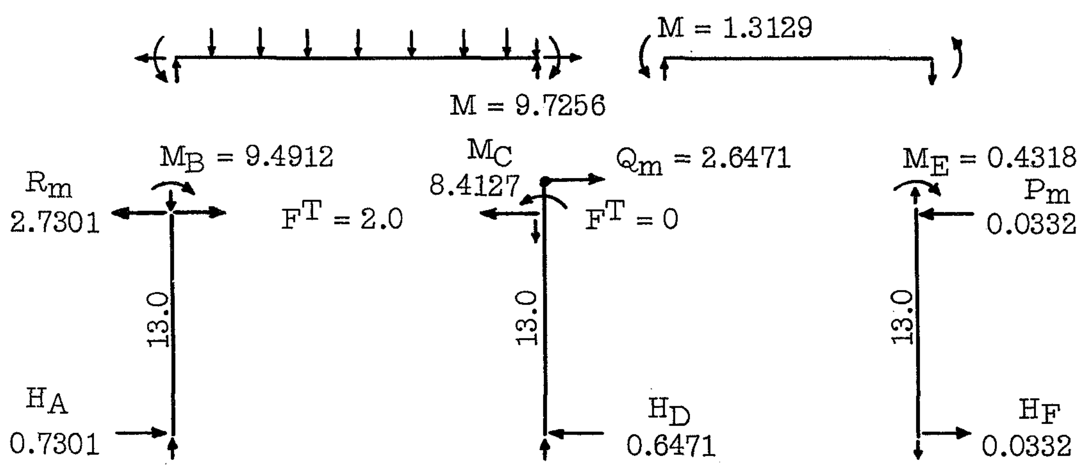
A complete description of the structure is given in figure 13(a). The truss is identical to the truss used in the preceding section XII-E. This example is also identical to the one used by Rogers[3], (p. 49). The truss constants are the same as given in section XII-E. From the equations in plates I and II, using numerical values from figure 13(a) the column constants for all three columns are



(a) Elevation of structure.



(b) Substitute structure and schematic moment distribution for step 1.



(c) Equilibrium diagram for step 1.

Figure 13.- Analysis of two bay Pratt trussed structure.

Moment stiffness $K_b = 2.5149$ AE

Moment due to shear stiffness $K_s = 0.19345$ AE

In this example actual numerical distribution was performed, the distribution being continued until the unbalance was insignificant in the fourth decimal place. However, in order to shorten the presentation and to simplify the figures, only the fixed end moments and the balanced moments are shown in the figures, along with the carryover factors and the distribution factors used in the analysis. The number of cycles of distribution required for convergence is noted in each step. It is assumed that the reader is familiar with the preceding examples. Therefore, some of the obvious procedure will not be redescribed in detail nor will some of the lengthy descriptive terms be repeated in full.

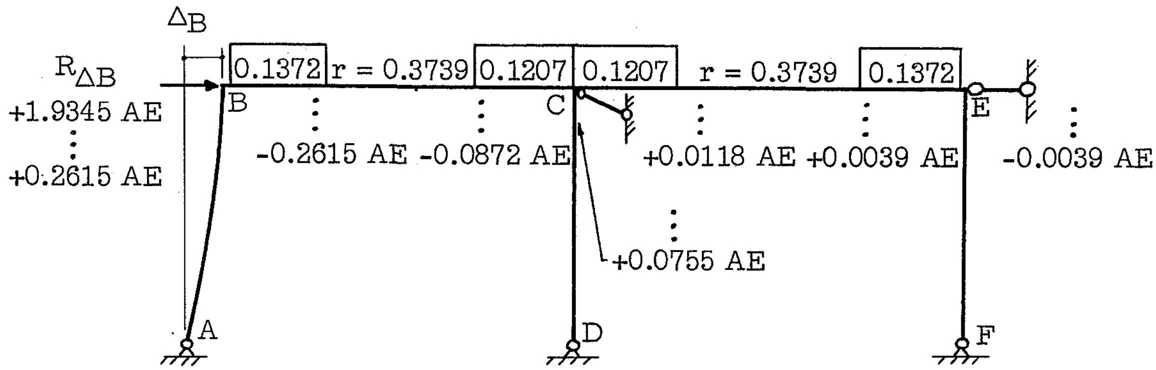
Step 1. The first step is to apply the truss fixed end moments and to balance the moments in the structure as indicated schematically in figure 13(b). An auxiliary force system is provided to prevent the translation of joints B, C, and E. Four cycles of distribution were sufficient. The equilibrium diagram is shown in figure 13(c), the column base shears and the auxiliary joint forces being determined by statics from the balanced moments.

Step 2. The second step is to translate joint B a distance Δ_B while preventing the translation of joints C and E. Again it is convenient to let $\Delta_B = 10$ units. The truss thrust becomes

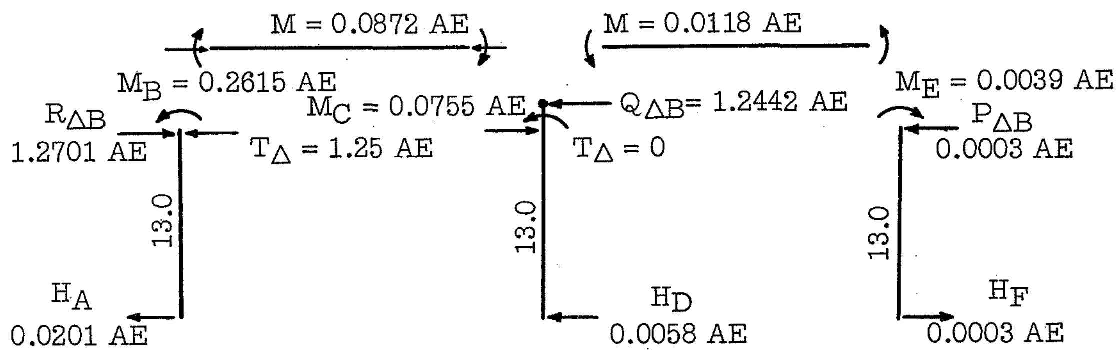
$$T_\Delta = 10\ K_t = 10(0.125\ AE) = 1.25\ AE$$

The fixed end moment in the column AB becomes

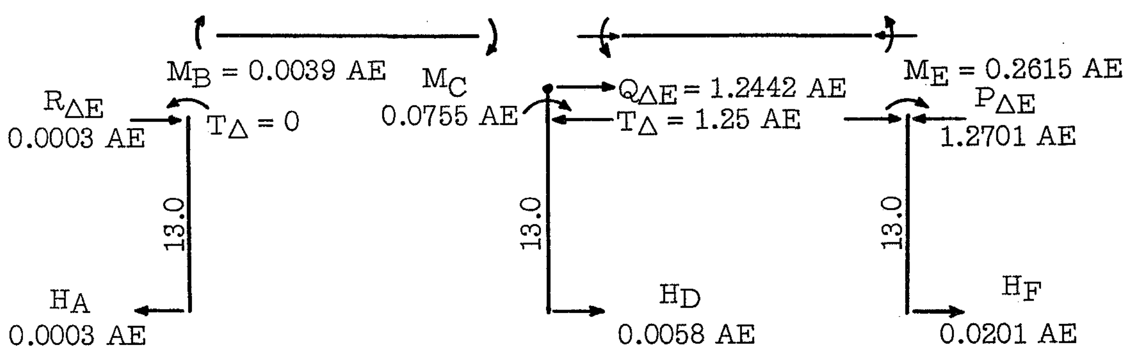
$$10\ K_s = 10(0.19345\ AE) = 1.9345\ AE$$



(d) Schematic moment distribution for step 2.



(e) Equilibrium diagram for step 2.



(f) Equilibrium diagram for step 3.

Figure 13.- Continued.

The moment distribution is performed as indicated schematically in figure 13(d). Three cycles of distribution were sufficient. The equilibrium diagram is shown in figure 13(e).

Step 3. The third step is to translate joint E a distance Δ_E while preventing the translation of joints B and C. Because the structure is symmetrical the force system in the equilibrium diagram for step 3 will be the mirror image of the force system in the equilibrium diagram of step 2 if $\Delta_E = \Delta_B = 10$ units. The equilibrium diagram for step 3 is shown in figure 13(f). If the structure is not symmetrical then a separate distribution must be performed for the effects of Δ_E.

Step 4. The fourth step is to translate joint C a distance $\Delta_C = 10$ units while preventing the translation of joints B and E. The translation in this case produces thrust in both trusses. The distribution of the fixed end moment in column DC is indicated in figure 13(g). Three cycles of distribution were sufficient. The equilibrium diagram is shown in figure 13(h).

Step 5. To obtain the final equilibrium state the external auxiliary forces must vanish. For these forces to vanish the following equations must be true:

$$-R_m + a\,R_{\Delta B} - b\,R_{\Delta C} + c\,R_{\Delta E} = 0$$

$$-Q_m - a\,Q_{\Delta B} + b\,Q_{\Delta C} + c\,Q_{\Delta E} = 0$$

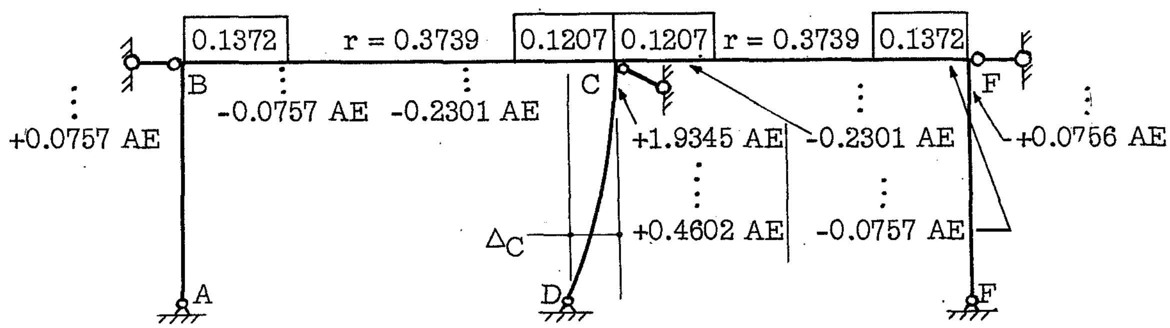
$$-P_m - a\,P_{\Delta B} - b\,P_{\Delta C} - c\,P_{\Delta E} = 0$$

Expressed in numerical values from figures 13(c), 13(e), 13(f), and 13(h) these thrust correction equations are

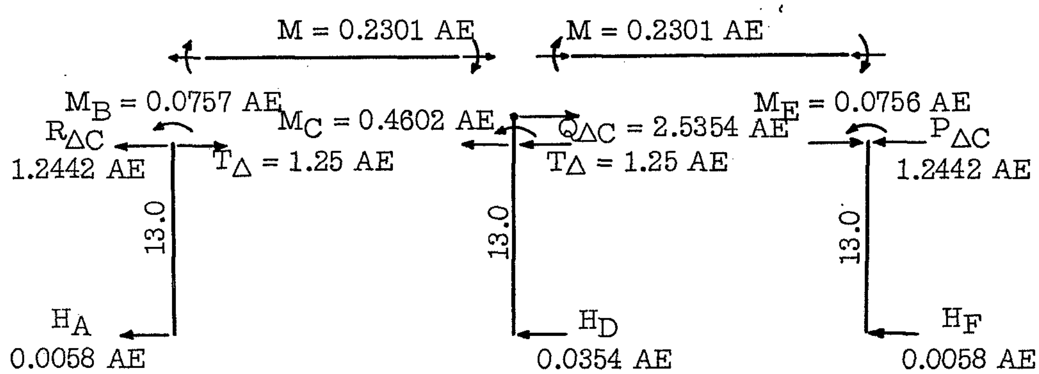
$$-2.7301 + 1.2701\,a - 1.2442\,b + 0.0003\,c = 0$$

$$2.6471 - 1.2442\,a + 2.5354\,b + 1.2442\,c = 0$$

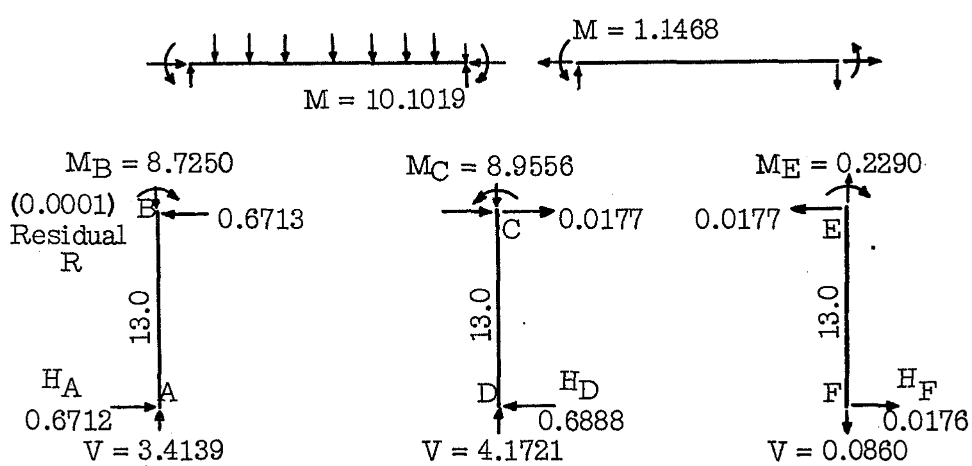
$$-0.0332 - 0.0003\,a - 1.2442\,b - 1.2701\,c = 0$$



(g) Schematic moment distribution for step 4.



(h) Equilibrium diagram for step 4.



(i) Final equilibrium diagram.

Figure 13.- Concluded.

The solution to these equations is

$$a = 2.7595/AE$$

$$b = 0.6225/AE$$

$$c = -0.6366/AE$$

The inclusion of the AE term was implied in writing the equations. The constant a is the correction factor by which the force system in figure 13(e) is multiplied; the constant b is the correction factor by which the force system in figure 13(h) is multiplied; and the constant c is the correction factor by which the force system in figure 13(f) is multiplied. These products are added to the force system of figure 13(c). The resultant sum of the respective moments and forces gives the final state of equilibrium, figure 13(i).

For comparison of results the column base shears are compared

	H_A	H_D	H_F
By present method	0.6712	0.6888	0.0176
By Least Work method	0.6714	0.6890	0.0176
	$\longrightarrow$	$\longleftarrow$	$\longrightarrow$

The Least Work solution for this case is not included here. However, the simultaneous equations obtained are given below for reference.

(1) $$2854.4\ H_A + 1427.2\ H_D + 832\ V_D = 4404$$

(2) $$1427.2\ H_A + 1494.4\ H_D + 416\ V_D = 1664$$

(3) $$832\ H_A + 416\ H_D + (350 + 16\sqrt{2})V_D = (1736 + 64\sqrt{2})$$

$$V_D = 4.1719$$

For the case of symmetrical loading of the above structure as presented in the following section G, the Least Work solution reduces to a set of simultaneous equations identical to those above except for the load terms on the right side of the equations. These load terms for the case of both trusses loaded are, in the same order

(1) 8808

(2) 4404

(3) $(3472 + 128\sqrt{2})$

From which

$$V_D = 8.3438$$

$$H_D = 0$$

G. Two bay structure - symmetric load

This structure is identical to the one in the preceding section XII-F. Both trusses are taken as completely loaded thus providing a symmetrical load on the structure. There is no need of a separate analysis because the equilibrium state for this example may be obtained by superimposing the force system of the final equilibrium diagram, figure 13(i), of the preceding example upon its mirror image. However, if a separate analysis were performed, it would not be necessary to make a thrust correction for the translation of joint C, because the structure and loading are symmetrical. One analysis could be made for the translation of joint B and its mirror image taken for an equal translation of joint E. Two equations would then be obtained, leading to the final state of equilibrium.

Before the Least Work solution for the preceding example was obtained in corrected form a separate numerical analysis was performed on this symmetrical case to check the analysis of the nonsymmetrical case. The results are presented below.

	H_A	H_D	H_F
By present method			
1. Separate analysis of symmetric loading	0.6536	0	0.6536
2. Nonsymmetric load analysis plus its mirror image	0.6536	0	0.6536
By Least Work method			
1. Nonsymmetric analysis plus its mirror image	0.6538	0	0.6538
2. Symmetric load analysis	0.6537	0	0.6537

XIII. THE APPLICATION TO FINK TRUSSED BENTS

A. Single bay bents

Two examples are shown to bring out several points in this type of truss and to provide reference figures for later discussion.

1. Symmetric bent.

The structure is shown in figure 14(a). The truss is identical to that in section X-C. The procedure for analysis is identical to that outlined in section XII-B. The only difference in this case is that δ is positive and also the fixed end thrust is of opposite sign than for the horizontal parallel chord truss. In general, this should be true for all arch-type trusses. Because the procedure has been outlined before only the final results are shown in figure 14(b).

By present method $\quad H_A = 0.2513$

By Least Work method $\quad H_A = 0.2513$

2. Nonsymmetric bent.

The structure is shown in figure 14(c). The procedure in this case is identical to that outlined in section XII-C. Only the final results are shown in figure 14(d).

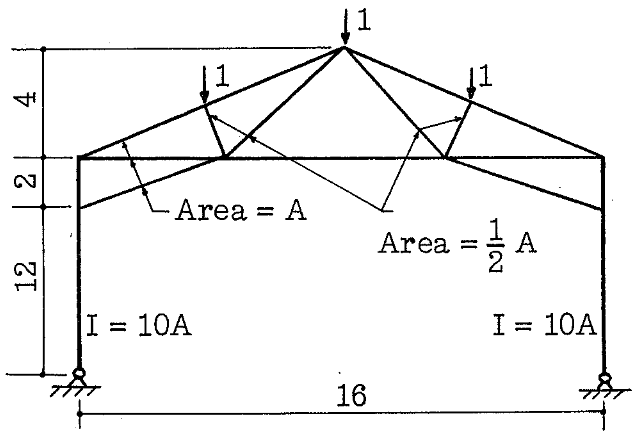
By present method $\quad H_A = 0.1710$

By Least Work method $\quad H_A = 0.1710$

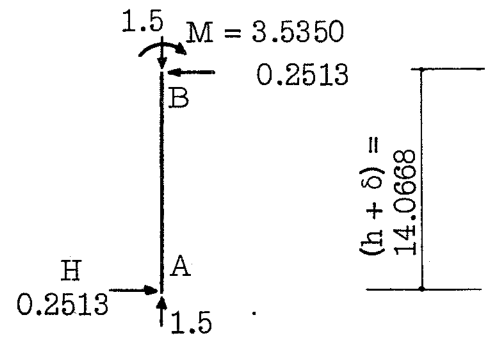
In each of the above cases the formulas developed in appendix A were used to obtain the moments.

B. Three bay symmetric structure

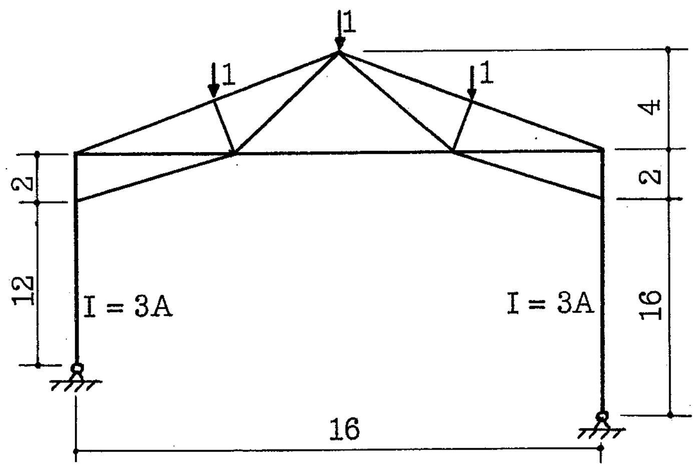
A description of the structure is given in figure 15(a). The truss is identical to the example in section X-C. The truss constants as evaluated in that section are



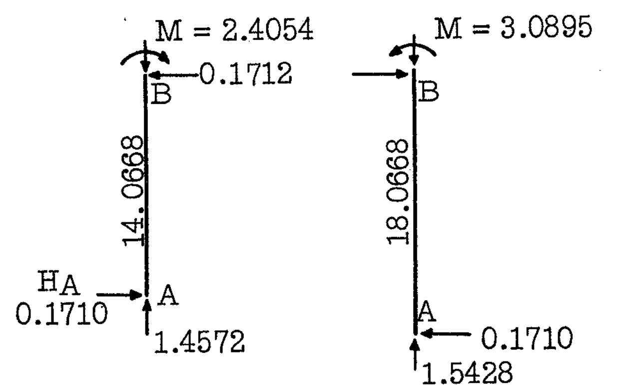
(a) Elevation of structure I.



(b) Final column equilibrium diagram.

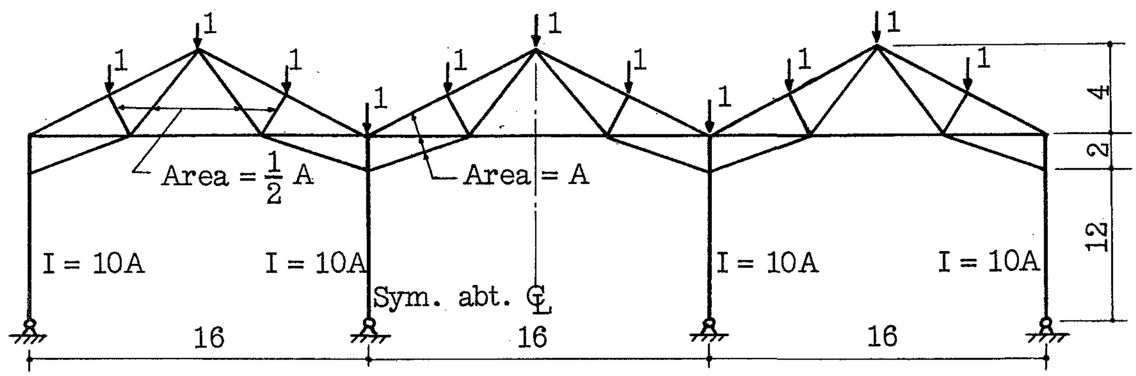


(c) Elevation of structure II.

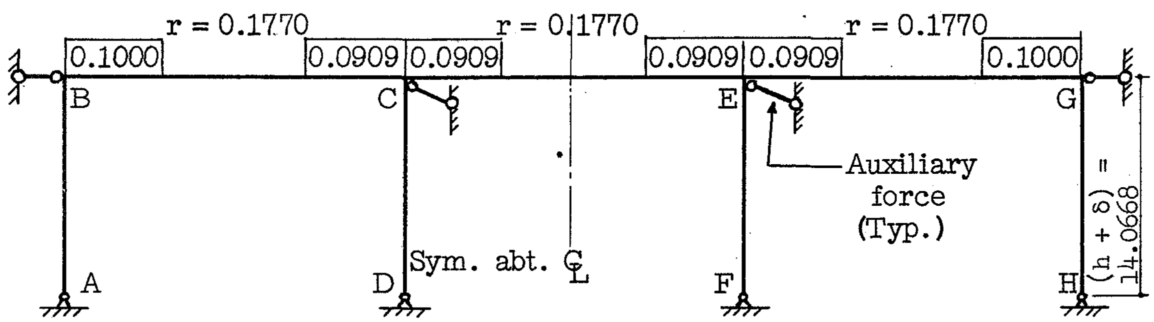


(d) Final column equilibrium diagram.

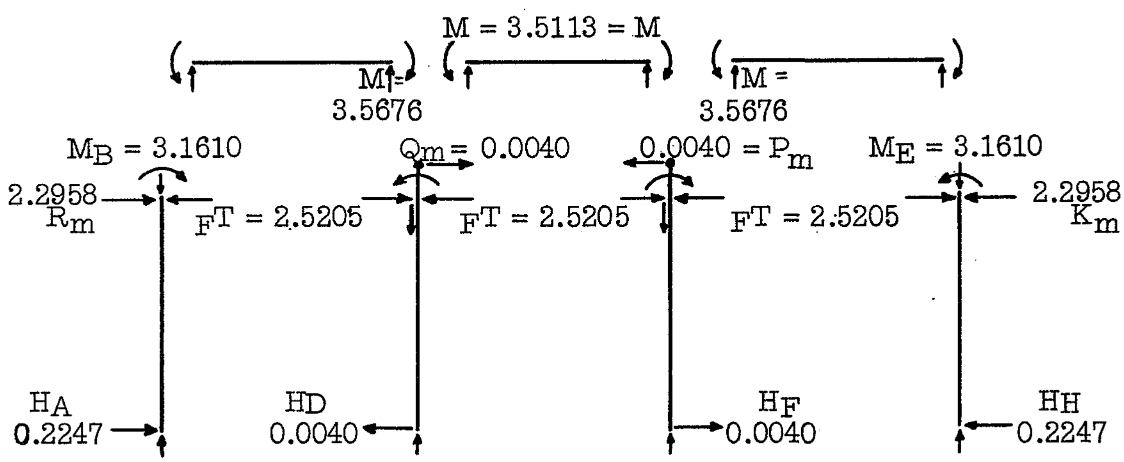
Figure 14.- Single bay Fink trussed structures.



(a) Elevation of structure.



(b) Substitute structure for moment distribution.



(c) Equilibrium diagram for step 1.

Figure 15.- Analysis of three bay Fink trussed structure.

Moment stiffness	$K_T = 0.3272$ AE
Thrust stiffness	$K_t = 0.062631$ AE
Carryover factor	$r = 0.1770$
Fixed end moment	$\mu M = 3.5112$
Fixed end thrust	$\mu T = 2.3205$
Location of "neutral point"	$\delta = 0.0668$ feet
	$(h + \delta) = 14.0668$ feet

From the equations in plates I and II, using the numerical values from figure 15(a), the column constants are

Moment stiffness	$K_c = 2.9446$ AE
Moment due to shear stiffness	$K_s = 0.2093$ AE

Because the structure and the load are symmetrical about an elevation center line several ways are possible to shorten the analysis and are used here. The extension of the presentation to a nonsymmetrical case should be obvious. Also, in order to shorten the presentation, the figures are presented in abbreviated form. The equilibrium diagrams only are shown for each step.

The fixed end moments for each case were applied to the substitute structure in figure 15(b) and the moments balanced numerically. Three cycles of distribution were sufficient in all cases to reduce the unbalance to a negligible amount in the fourth decimal place. The column base shears and the auxiliary forces were computed by statics. To simplify the diagrams the vertical reactions are not shown for all steps. They are indicated with values only in the final equilibrium diagram.

Step 1. Figure 15(c) shows the equilibrium diagram for the vertical load analysis with joints B, C. E, and G held against translation.

Step 2. Joint B is translated a distance $\Delta_B = 10$ units with joints C, E, and G held against translation. Figure 15(d) shows the equilibrium diagram. The equilibrium diagram for joint G translated $\Delta_G = -10$ units is the mirror image of figure 15(d) and is not shown here.

Step 3. Joint C is translated a distance $\Delta_C = 10$ units with joints B, E, and G held against translation. Figure 15(e) shows the equilibrium diagram. The equilibrium diagram for joint E translated $\Delta_E = -10$ units is the mirror image of figure 15(e) and is not shown.

The auxiliary forces R_m, Q_m, F_m, and K_m must be made to vanish. From the equilibrium diagrams of the above three steps the thrust correction equations for the elimination of the auxiliary forces become in numerical values

$$\begin{aligned}
2.2958 + 6.4117\,a - 6.2415\,b + 0.0003\,c &= 0 \\
0.0040 - 6.2416\,a + 12.7962\,b + 6.2435c + 0.0002\,d &= 0 \\
-0.0040 - 0.0002\,a - 6.2435\,b - 12.7962c + 6.2416\,d &= 0 \\
-2.2958 - 0.0003\,b + 6.2415\,c - 6.4117\,d &= 0
\end{aligned}$$

Because the structure and the load are symmetrical in this case the solution of the equations was simplified by recognizing that

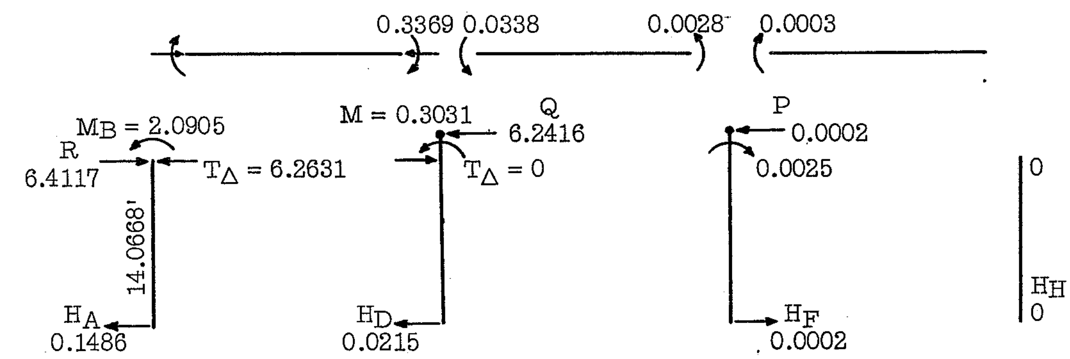
$$a = d \quad \text{and} \quad b = c$$

The first two equations were used, giving the solution as

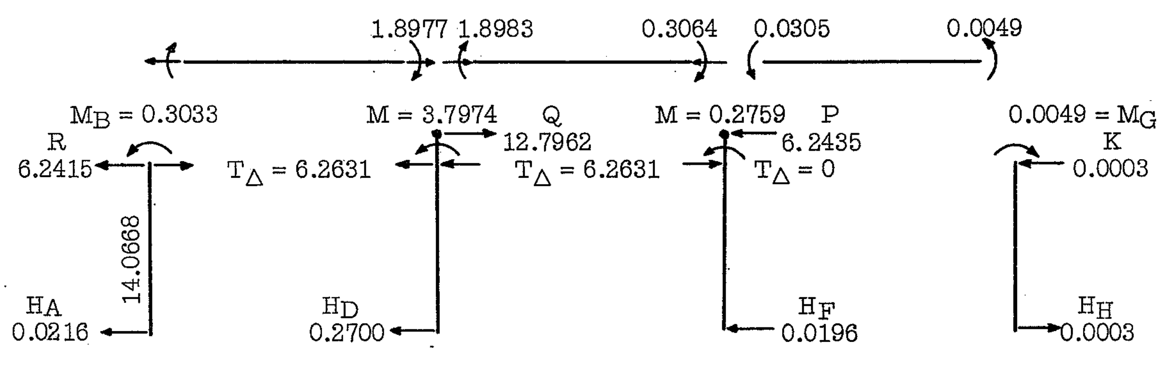
$$a = d = -0.5262/AE$$

$$b = c = -0.1727/AE$$

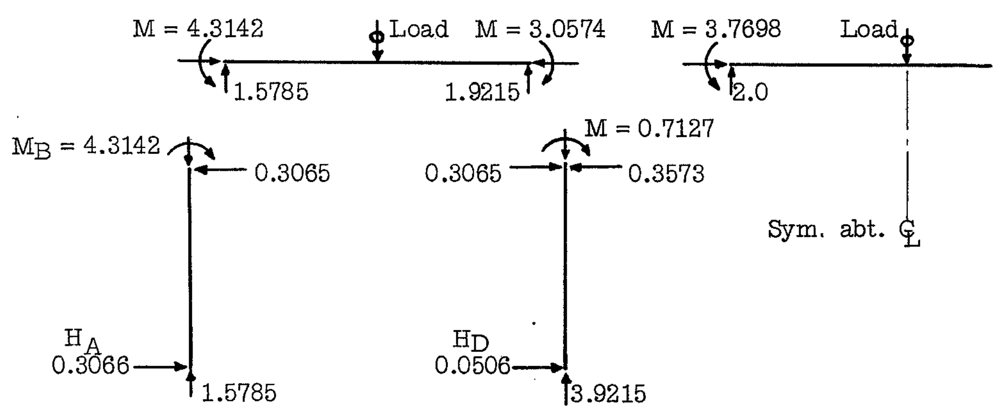
The inclusion of the AE term was implied in writing the equations. The force system in figure 15(d) is multiplied by the constant a and the force system in the mirror image of figure 15(d) is multiplied by the constant d. The force system in figure 15(e) is multiplied by the



(d) Equilibrium diagram for Δ_B = 100 units/AE.



(e) Equilibrium diagram for Δ_C = 100 units/AE.



(f) Final equilibrium diagram.

Figure 15.- Concluded.

constant b and the force system in the mirror image of figure 15(e) is multiplied by the constant c. The products are added to the force system in figure 15(c). The resultant sum of the respective forces and moments is the final state of equilibrium, figure 15(f). The vertical reactions were computed from the final equilibrium diagram.

The Least Work solution to this example was made by Vitagliano[4]. However, by the author's own admission, his results are in error. The results of this solution are reproduced here but it should not be inferred that the apparent differences in the values actually exist between the present method and the correct Least Work solution. It was not considered worthwhile to reproduce the Least Work solution to obtain better agreement. On the basis of results from previous examples the reactions obtained by the present method should not differ from the correct Least Work solution by more than 0.1 percent. For further comparison the solution obtained by Brown[5] is also included. The method used by Brown is also convergent but the convergence is slow.

	H_A	H_D	V_D
By present method	0.3066	0.0506	3.9215
Solution by Vitagliano[4] (Least Work method)	0.3281	0.0603	3.9145
Solution by Brown[5]	0.3109	0.0555	3.9141

XIV. DISCUSSION OF RESULTS

The results obtained by use of the present method are in excellent agreement with the results obtained by the Least Work method. In order to obtain accurate results for comparative purposes the computations were carried to four decimal places with original numbers of five and six significant figures in the constants of the structure. Also certain truss constants were sometimes manipulated to obtain significant figures in the column base shears at important points. Three significant figures in the column base shears, where possible, were generally considered a sufficient minimum, although four significant figures were generally used in the simple structures. In multibay structures it is not feasible to hold to a given number of significant figures for all resulting values because the moments (and the column base shears) decay rapidly away from the point of application of the fixed end moments. This is obvious in figures 13(c), 13(e), 13(h) and 15(d) and 15(e).

Also of interest is how much error can be expected in the results of an analysis if the truss thrusts are entirely neglected. Comparisons can be made for the specific examples presented herein if for the no thrust condition the present method of analysis is used to obtain the balanced moments and base shears due to the vertical loads.

For illustration consider the first example of the parallel chord Pratt truss, section XII-B, and the associated figures 8(c) and 8(f). If the thrust is neglected then the results from the distribution of moments in the first step, figure 8(c), will provide the solution because the fixed end thrust is considered zero.

Correct solution	$H_A = 0.3011$
Solution for no thrust	$H_A = 0.3253$
Error	8 percent

Now consider the example of the Fink truss in section XIII-A, 1, and the associated figures 14(a) and 14(b). The moment distribution for the first step is not shown here but the results are used.

Correct solution	$H_A = 0.2513$
Solution for no thrust	$H_A = 0.2287$
Error	-9.9 percent

For this particular Fink trussed bent the error in H_A due to neglect of thrust is approximately -10 percent for variations in the column I's from $I = 2A$ to $I = 100A$.

In the example of the two bay structure in section XII-G, the column base shears are about 4 percent higher than the exact solution if the truss thrusts are neglected.

Table D is a presentation of comparative errors for other examples presented herein. For the case of the eight-panel Pratt truss with fixed end column bases it is evident that the neglect of truss thrust can result in considerable error in the column base shear and base moment although the moment in the column at the lower end-panel point of the truss is not affected to the same degree.

Figure 16 shows the column base moment and base shear variation for different column I values for the eight-panel Pratt truss with fixed end columns. It is noted that the base moment M_A changes sign at approximately $I = 26A$. Also the value of H_A diminishes after a peak value at approximately $I = 4A$. These results are not indicated in an analysis where the truss thrust is neglected.

TABLE D.- TABLE OF COMPARATIVE ERROR FOR NEGLECT OF TRUSS THRUST

1. Nonsymmetric Pratt Truss in Section XII-C, Figure 9(a).

Column I	3A	10A	100A
Percent error in H	6.4	6.7	7

2. Nonsymmetric Fink Truss, Figure 14(c).

Column I	3A	10A	20A	100A
Percent error in H	-8.2	-8.3	-8.0	-8.2

3. Symmetric Eight-Panel Pratt Truss in Section XII-E, Figure 11(a).

Column I	4A	8A	10A	20A	25A	40A	100A
Percent error in H_A	2.5	9.0	12.4	31.5	43	87	196
Percent error in M_A	28	61	84	450	7900	wrong sign	
Percent error in M^*	-18.9	-2.4	3.2	-8.9	-7.9	-8.0	-8.7

$M^* = H_A h_1 + M_A$

Positive error is higher than, and negative error is lower than, the correct value.

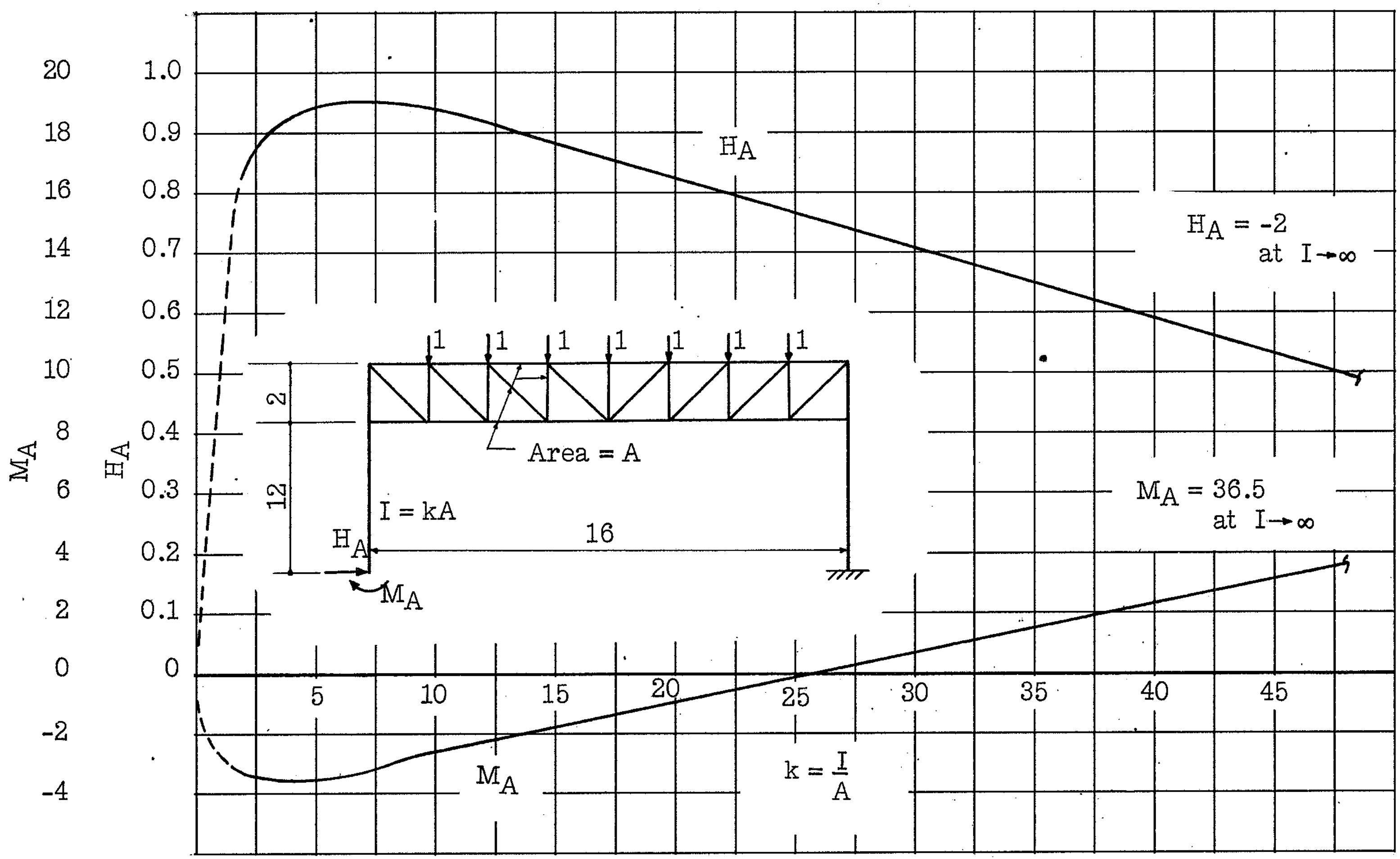


Figure 16.- Column base moment and base shear variation for symmetric eight-panel Pratt truss with fixed end columns.

In the three bay Fink trussed structure in section XIII-B, the results of neglecting thrust are

	H_A	H_D	V_D
Correct solution	0.3066	0.0306	3.9213
Solution for no thrust	0.2247	-0.0040	4.0254
Error	-27 percent		2.7 percent

A preliminary investigation was made for a more realistic single bay structure consisting of a 14-panel, 42- by 3-foot deep parallel chord Pratt truss with ten WF 49 columns of clear height 18 feet and hinged column bases. The structure represented roughly a balanced design for ordinary loading conditions. The column I was approximately equal to 34 A where A is the chord area only. The column base shear obtained by neglecting the thrust was about 7 percent higher than the exact value.

It appears that for single bay structures with hinged column bases the column base shear obtained by neglecting the truss thrust is, for vertical loadings: 1. Slightly greater than the exact solution for horizontal parallel chord trusses; 2. Smaller than the exact solution for Fink trusses or arched type structures. However, for fixed column bases the neglect of thrust can result in serious error as illustrated in table D-3. No other generalizations are warranted on the basis of the examples worked here.

XV. CONCLUSIONS

1. That with the method presented moment distribution can be easily applied to trussed bents with assurance of the correctness of method.

2. That it can be shown that the present method converges to the Least Work solution.

3. That because the derivations of the truss constants were made in general form the method and the equations derived may be assumed to hold true for all types of trusses that may be used for trussed bents.

4. That the method requires little more work than the analysis of a solid member bent provided the truss constants are known.

5. That, in general, the convergence of the moments is rapid. This is due to the small carryover factor and the small distribution factors for the truss. This should be true for the majority of practical cases.

6. That the analysis of multibay structures can be made with relative ease in comparison with the classical methods, even if the truss constants must be determined. However, there is little or no saving in work over the Least Work method for a single bay structure unless the truss constants are known.

7. That sidesway due to nonsymmetric structure or nonsymmetric load is taken care of automatically by the method. This occurs in the thrust correction equations.

8. That the standardization of trusses and subsequent tabulation of truss constants are now both highly feasible and desirable. The tabulation of truss constants is feasible because the constants can now be referred to a single point - the "neutral point" defined herein.

9. The only apparent objectionable feature of the present method is the necessity of solving simultaneously the thrust correction equations. There will be one more equation than the number of bays except for the case of symmetric structure and load. These equations must be solved on a calculating machine because, in general, small differences of large numbers are involved. The use of slide rule is not recommended.

10. Because the examples presented were largely academic no study was made to determine just how many significant figures are required for various degrees of accuracy of the solution. Four significant figures in the structure constants should be sufficient. Of course decimal point manipulations may be necessary in certain cases to obtain significant figures in the column base shears. However, the use of excessive figures, say more than six, can result in mistakes in using the calculator.

11. No general recommendations are offered as for what cases the thrust may be safely neglected in the analysis. Any worthwhile recommendations would require extensive investigation for many different cases and is beyond the scope of this work. Therefore, on the basis of the results obtained herein, it is recommended at the present time that the effects of thrust be considered in all cases, especially multibay structures.

XVI. SUMMARY

The method of application of moment distribution to trussed bents developed in this investigation provides an accurate and relatively easy method of analysis of trussed bents. It is believed that the method fills a gap that has long been overlooked in the many applications of moment distribution to structural analysis.

A point at the end of the truss about which moments may be varied without changing the thrust and at which the thrust may be varied without changing the moment is located and defined as the "neutral point" of the truss. Specific equations to express the location of the "neutral point" and the necessary truss constants are developed in general terms of the truss stress summations and truss dimensions. A substitute column-truss attachment is devised as a concentrated equivalent joint at the "neutral point." Equations are developed to express the necessary constants of the substitute column relative to the "neutral point."

These innovations provide a substitute structure to which moment distribution may be applied in the ordinary manner. Also this substitute structure provides a means of incorporating the trussed bent into a more complicated structure such as a trussed bent within or on top of a multi-story rigid frame structure. The inclusion of foundation rotations or translations is made in the same manner as for solid member bents.

The many examples presented should thoroughly illustrate the method and acquaint the reader with the application. Obviously there may be many structures which are not duplications of the types presented herein but the material presented should provide the basis for extension of the method to handle these particular cases.

XVII. ACKNOWLEDGEMENTS

The author wishes to express his appreciation to the National Aeronautics and Space Administration, Langley Research Center, for the opportunity to write this thesis and for the reproduction of the material.

He also wishes to thank Professor D. H. Pletta, Head, Department of Engineering Mechanics, Virginia Polytechnic Institute, for his suggestions and constructive criticism during the preparation of this thesis.

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* M.S. Thesis in Architectural Engineering

** M.S. Thesis in Engineering Mechanics

XIX. VITA

Henry T. Thornton, Jr. was born in Pasquotank County, North Carolina, September 13, 1927. He attended various public and private schools in North Carolina, graduating from Elizabeth City High School, Elizabeth City, North Carolina, in 1944. He served in the U. S. Maritime Service and sailed in the Merchant Marine in the Pacific. He served in the U. S. Army with a tour of duty in Italy, being honorably discharged in June 1947. He attended Gardner Webb Junior College and was awarded the Degree of Bachelor of Civil Engineering by North Carolina State College in June 1951. He was associated with the National Advisory Committee for Aeronautics from 1951 until 1958 and has been associated with the National Aeronautics and Space Administration since 1958. He is a registered professional civil engineer in Virginia. Langley Research

XX. APPENDICES

APPENDIX A. Moment Distribution by the Sum of a Series

A. General

In order to save calculation time and to provide for more accurate results in the analysis for comparative purposes, series solutions were devised to express the balanced moments obtained by moment distribution for the single bay structure and for the particular loadings encountered in the investigation. The method of derivation will be illustrated for one particular loading on a generalized bent and only the final results will be presented for the other cases.

In the derivations it is assumed that under an axial load of the magnitude of the bent base reactions the beam will shorten or elongate a relatively large amount. This is evident by the use of auxiliary restraining forces at each end of the beam. This was done to make the derivations adaptable to the trussed bents used in the main presentation. Also the beam is considered physically symmetrical about its elevation center line so that the moment stiffness and carryover factor are the same for both ends. Because there are no carryover moments from the column bases the equations are simplified greatly.

In order to eliminate excessive subscripts the balanced moments carry only one subscript denoting the joint at which the moment acts (i.e., M_B). Fixed end moments are given as $_FM_B$, denoting the fixed end moment at joint B. The origin of the fixed end moments and whether they originate in the column or in the beam is designated by word or other indication but not by subscript.

B. Sign convention

A moment is considered positive if it tends to rotate the adjacent joint clockwise. All fixed end moments have been assumed positive. For a negative fixed end moment it is necessary only to insert the moment with correct sign in the general equation. All equations for the balanced moment at the column-beam joint give the beam moment with correct sign. This form was chosen for conformity and because the sign is identical to the sign of the moment to be applied to the column as a free body.

C. Column base detail

All cases have been derived for fixed column bases, and the value of the column moment stiffness used in the moment equations must of course be for the fixed end column. With certain restrictions the equations apply also for the case of hinged column bases. The column base moment of course becomes zero. Because there is no carryover from the base to the column-beam joint in the derivations the general moment equations for the column-beam joint are not changed. Therefore, it is necessary only to use for the column moment stiffness in the equations its value for the case of the hinged column base.

D. Illustration of derivation

Figure A-1 shows the distribution and balancing of moments in arbitrary terms for beam fixed end moments due to a vertical beam load. The column-beam joints are held against translation. The beam distribution factors are γ and β.

$$\gamma = \frac{K_{C1}}{K_{C1} + K_T} \; ; \quad \beta = \frac{K_{C2}}{K_{C2} + K_T}$$

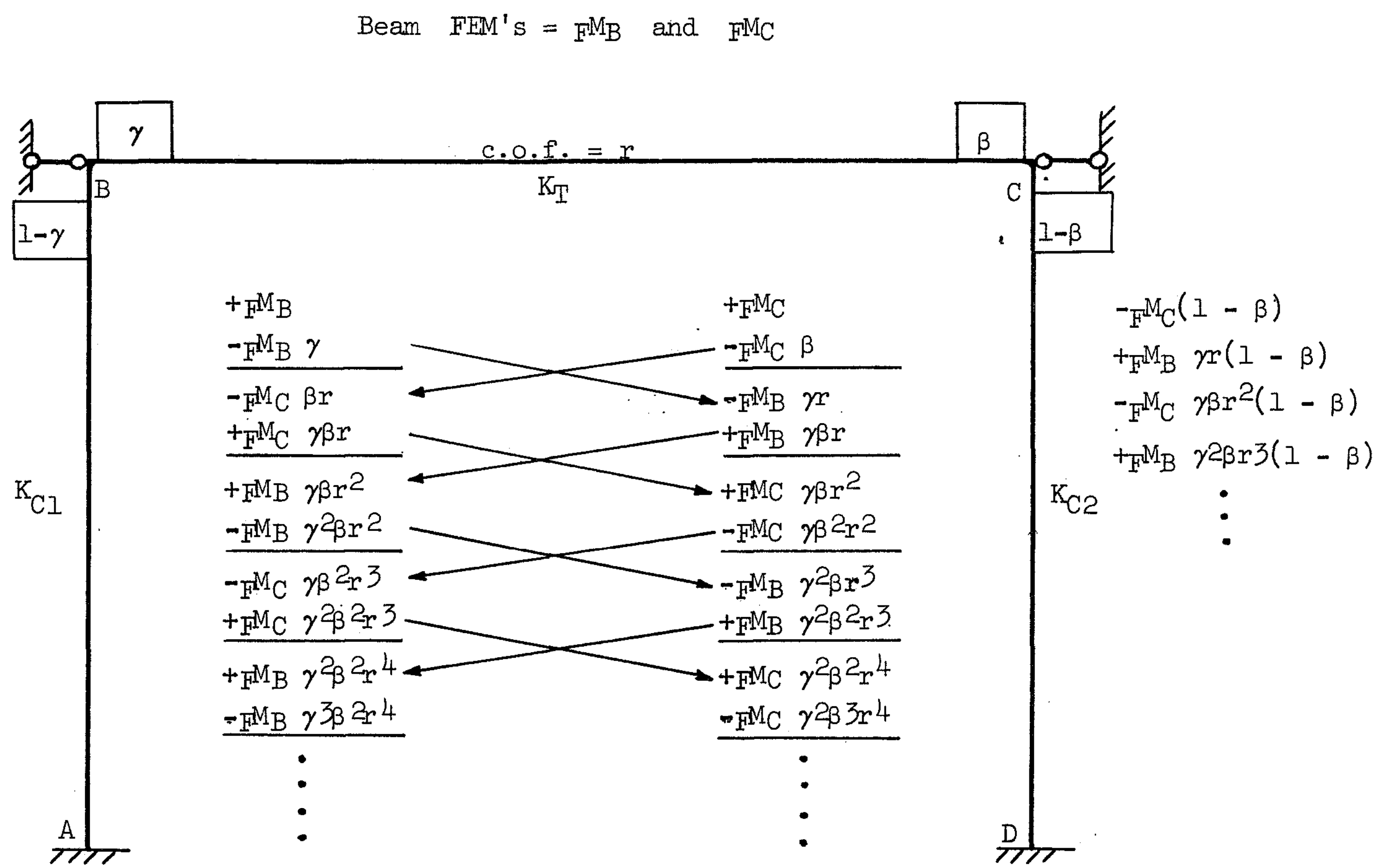


Figure A-1.- Moment distribution in general terms.

Summing the total series of terms for the beam moment at B

$$M_B = {}_F M_B(1-\gamma)\left[1+\gamma\beta r^2+\gamma^2\beta^2 r^4+\gamma^3\beta^3 r^6+\dots\right] +$$
$$- {}_F M_C(1-\gamma)\beta r\left[1+\gamma\beta r^2+\gamma^2\beta^2 r^4+\gamma^3\beta^3 r^6+\dots\right]$$

Both γ and β are less than unity. The beam carryover factor r is assumed less than unity. Therefore, the series within the brackets is convergent. Its n^{th} term will be

$$(\gamma\beta)^{n-1}\, r^{2n-2} = (\gamma\beta r^2)^{n-1}$$

And the common ratio is

$$\frac{1}{\gamma\beta r^2}$$

Therefore the sum of the series is

$$\text{Sum} = \frac{1}{1-\gamma\beta r^2} \qquad \gamma\beta r^2 < 1$$

The balanced moment then becomes

$$M_B = ({}_F M_B - \beta r\, {}_F M_C)\,\frac{(1-\gamma)}{1-\gamma\beta r^2}$$

Similarly

$$M_C = ({}_F M_C - \gamma r\, {}_F M_B)\,\frac{(1-\beta)}{1-\gamma\beta r^2}$$

The column moment at B is of course the negative of M_B given above. For this particular case the column base moment at A is

$$M_A = r_c(-M_B)$$

The above equations are in general terms of the distribution factors. In cases of symmetrical load and symmetrical structure the equations may be further simplified. Let

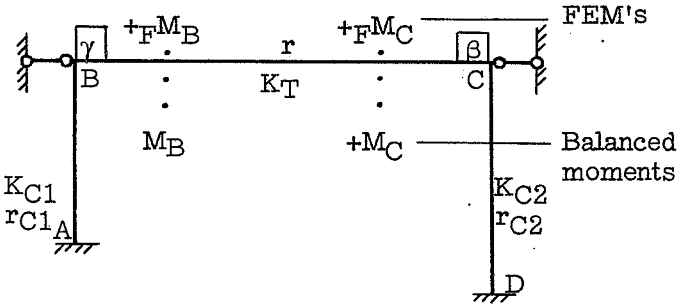
$$pM = pM_B = -pM_C; \quad \text{and} \quad K_{C1} = K_{C2}$$

Then

$$M_B = pM\left[1 + \frac{K_T}{K_C + K_T}\,r\right]\left[1 - \frac{K_T}{K_C + K_T}\right]\frac{1}{\left[1 - \left(\frac{K_T}{K_C + K_T}\right)^2 r^2\right]}$$

$$= pM\,\frac{K_C}{K_C + K_T(1 - r)}$$

A. Nonsymmetric structure - nonsymmetric beam load.



$$M_B = ({}_FM_B - \beta r\ {}_FM_C)\ \frac{(1 - \gamma)}{1 - \gamma\beta r^2} \qquad (2.01)$$

$$M_C = ({}_FM_C - \gamma r\ {}_FM_B)\ \frac{(1 - \beta)}{1 - \gamma\beta r^2} \qquad (2.02)$$

$$M_A = - r_{C1}\ M_B \qquad (2.03)$$

$$M_D = - r_{C2}\ M_C \qquad (2.04)$$

B. Nonsymmetric structure- symmetric beam load.

$${}_FM = {}_FM_B = -{}_FM_C$$

$$M_B = {}_FM\ \frac{K_{C1}\left[(K_{C2} + K_T) + rK_T\right]}{(K_{C1} + K_T)(K_{C2} + K_T) - (rK_T)^2} \qquad (2.05)$$

$$M_C = {}_FM\ \frac{K_{C2}\left[K_{C1} + K_T + rK_T\right]}{(K_{C1} + K_T)(K_{C2} + K_T) - (rK_T)^2} \qquad (2.06)$$

C. Symmetric structure - symmetric beam load.

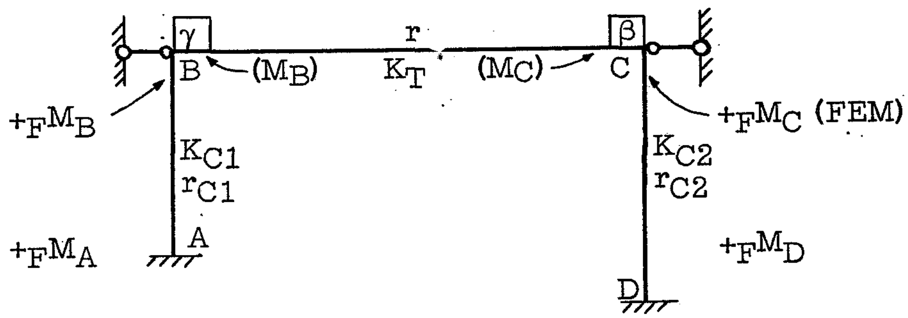
$${}_FM = {}_FM_B = -{}_FM_C; \quad K_{C1} = K_{C2}$$

$$M_B = -M_C = {}_FM\ \frac{K_C}{K_C + K_T(1 - r)} \qquad (2.07)$$

(a) Case I.

Figure A-2.- Equations for balanced moments.

A. Nonsymmetric structure - general column loads.



Balanced moments

$$M_B = \left[-{}_FM_B(1 - \beta r^2)\gamma - {}_FM_C(1 - \gamma)\beta r\right]\frac{1}{1 - \gamma\beta r^2} \tag{2.08}$$

$$M_C = \left[-{}_FM_C(1 - \gamma r^2)\beta - {}_FM_B(1 - \beta)\gamma r\right]\frac{1}{1 - \gamma\beta r^2} \tag{2.09}$$

$$M_A = {}_FM_A + r_{C1}(-{}_FM_B + {}_FM_C\ \beta r)\ \frac{(1 - \gamma)}{1 - \gamma\beta r^2} \tag{2.10}$$

$$M_D = {}_FM_D + r_{C2}(-{}_FM_C + {}_FM_B\ \gamma r)\ \frac{(1 - \beta)}{1 - \gamma\beta r^2} \tag{2.11}$$

B. Symmetric structure - symmetric column loads.

$${}_FM = {}_FM_B = -{}_FM_C;\ K_{C1} = K_{C2}$$

$$M_B = -M_C = -{}_FM\ \frac{K_T(1 - r)}{K_C + K_T(1 - r)} \tag{2.12}$$

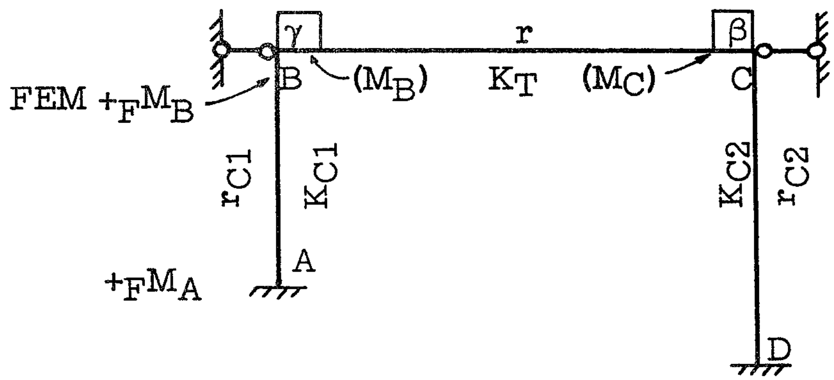
$$M_A = {}_FM_A - r_{C1}\ {}_FM\ \frac{K_C}{K_C + K_T(1 - r)} \tag{2.13}$$

$$M_D = {}_FM_D + r_{C2}\ {}_FM\ \frac{K_C}{K_C + K_T(1 - r)} \tag{2.14}$$

(b) Case II.

Figure A-2.- Continued.

A. Nonsymmetric structure - column load one side only.



Balanced moments

$$M_B = -{}_FM_B \frac{\gamma(1 - \beta r^2)}{(1 - \gamma\beta r^2)}$$

$$= -{}_FM_B \frac{K_T\left[K_{C1} + K_T - r^2K_T\right]}{(K_{C1} + K_T)(K_{C2} + K_T) - (rK_T)^2} \qquad (2.15)$$

$$M_C = -{}_FM_B \frac{\gamma r(1 - \beta)}{1 - \gamma\beta r^2}$$

$$= -{}_FM_B \frac{K_{C2}\, rK_T}{(K_{C1} + K_T)(K_{C2} + K_T) - (rK_T)^2} \qquad (2.16)$$

$$M_A = {}_FM_A - r_{C1}\, {}_FM_B \frac{(1 - \gamma)}{(1 - \gamma\beta r^2)}$$

$$= {}_FM_A - r_{C1}\, {}_FM_B \frac{K_{C1}\,(K_{C2} + K_T)}{(K_{C1} + K_T)(K_{C2} + K_T) - (rK_T)^2} \qquad (2.17)$$

$$M_D = r_{C2}\, {}_FM_B \frac{\gamma r(1 - \beta)}{1 - \gamma\beta r^2}$$

$$= -r_{C_2}\, M_C \qquad (2.18)$$

(c) Case III.

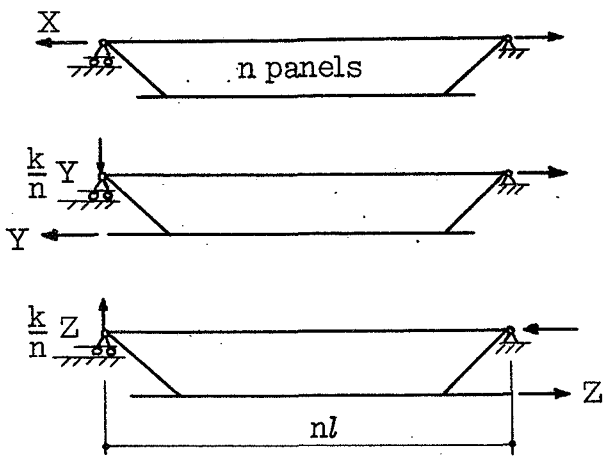
Figure A-2.- Concluded.

APPENDIX B. Supplementary Data on Parallel Chord Trusses

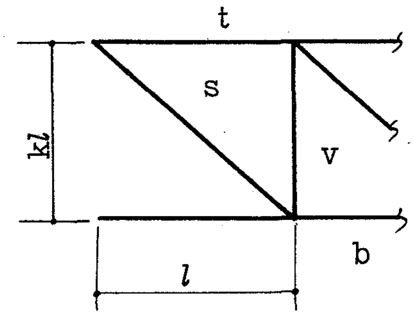
A. General

In the course of the investigation of trussed bents it became desirable to express the truss stress summations for the n-paneled Pratt truss in general terms of the number of panels, the truss geometry and the member areas. These expressions are shown in figures B-1 and B-2. The summations are of course for the restricted case of constant area chords and for all of the diagonals of equal area and all of the verticals of equal area. Any further generalization of the member sizes would not be worthwhile.

The general summations were obtained by making stress calculations for 4, 6, 8, 10, and 12 panel trusses and tabulating the various component parts such as top chord stress for a given summation. From these tabulations the repetitive pattern of increase was determined and the summation of the particular part expressed in terms of the number of panels n. The sums occur as the sum of the first p numbers, the sum of the squares of the first p numbers, etc.



Truss loads



Panel nomenclature

S = member stress

$\lambda = \sqrt{1 + k^2}$

Summations for $X = Y = Z = 1$.

$$\sum S_X^2 \frac{L}{A} = n \frac{l}{t}$$

$$\sum S_X S_Y \frac{L}{A} = \frac{n}{2} \frac{l}{t}$$

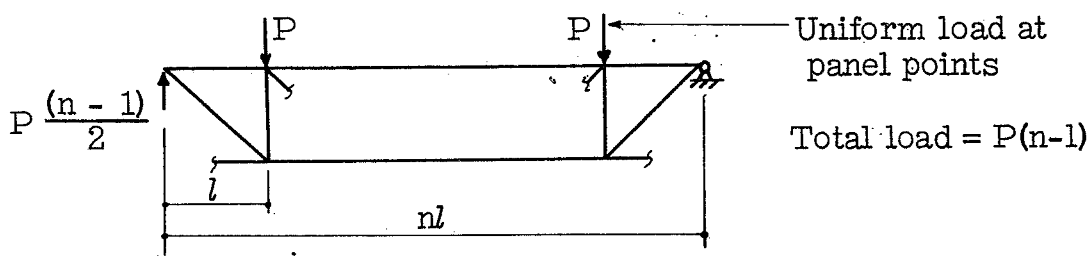
$$\sum S_X S_Z \frac{L}{A} = - \frac{n}{2} \frac{l}{t}$$

$$\sum S_Y^2 \frac{L}{A} = \left[\frac{n(4n^2 - 3n + 2)}{12n^2} \frac{l}{t} + \frac{n}{n^2} \lambda^3 \frac{l}{s} + \frac{(n - 2)}{n^2} k^3 \frac{l}{v} + \frac{n(4n^2 + 3n + 2)}{12n^2} \frac{l}{b}\right]$$

$$\sum S_Z^2 \frac{L}{A} = \sum S_Y^2 \frac{L}{A}$$

$$\sum S_Y S_Z \frac{L}{A} = - \left[\frac{n(4n^2 - 3n + 2)}{12n^2} \frac{l}{t} + \frac{n}{n^2} \cdot \lambda^3 \frac{l}{s} + \frac{(n - 2)}{n^2} k^3 \frac{l}{v} - \frac{n(2n^2 - 3n - 2)}{12n^2} \frac{l}{b}\right]$$

Figure B-1.- Truss stress summations for n-paneled Pratt truss for general end loads.



Truss loads

For panel nomenclature see figure B-1.

Stress summations.

$$\sum S_P\, S_X\, \frac{L}{A} = -\,\frac{n(2n^2 + 3n - 2)}{24}\,\frac{P}{k}\,\frac{l}{t}$$

$$\sum S_P\, S_Y\, \frac{L}{A} = -\left[\frac{n(n + 2)(2n - 1)}{48}\,\frac{l}{t} - \frac{n(n - 2)(2n + 1)}{48}\,\frac{l}{b}\right]\frac{P}{k}$$

$$\sum S_P\, S_Z\, \frac{L}{A} = \left[\frac{n(n + 2)(2n - 1)}{48}\,\frac{l}{t} + \frac{n(n - 2)(2n + 1)}{48}\,\frac{l}{b}\right]\frac{P}{k}$$

Figure B-2.- Truss stress summations for n-paneled Pratt truss for uniform panel point load.